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Structure of Divergence-Free Lie Algebras

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Abstract

One of the four well-known series of simple Lie algebras of Cartan type is the series of Lie algebras of Special type, which are divergence-free Lie algebras associated with polynomial algebras and the operators of taking partial derivatives, connected with volume-preserving diffeomorphisms. In this paper, we determine the structure space of the divergence-free Lie algebras associated with pairs of a commutative associative algebra with an identity element and its finite-dimensional commutative locally-finite derivation subalgebra such that the commutative associative algebra is derivation-simple with respect to the derivation subalgebra.

1 Introduction

Volume-preserving diffeomorphisms are important transformations on manifolds. Lie groups of volume-preserving diffeomorphisms are fundamental structures in the theory of idealized fluids. They have been studied through their Lie algebras, whose elements are divergence-free. One of the four well-known series of simple Lie algebras of Cartan type is the series of Lie algebras of Special type, which are divergence-free Lie algebras associated with polynomial algebras and the operators of taking partial derivatives. Graded generalizations of the Lie algebras of Special type have been studied by Kac [K1], Osborn [O], Djokovic and Zhao [DZ], and Zhao [Z]. Supersymmetric graded generalizations of the algebras have been investigated by Kac [K2], [K4]. Motivated from the nongradedness of the Lie algebras generated by conformal algebras (cf. [K3], [X2]) and the Lie algebras associated with vertex operator algebras (cf. [B], [FLM]), the second author [X1] introduced a natural class of nongraded generalizations of the Lie algebras of Special type, which are simple algebras in general. The term "divergence-free" is used because we want to give the readers an intuitive picture of these Lie algebras. In this paper, we shall determine the isomorphism classes of the divergence-free Lie algebras introduced in [X2]. Below we shall give a detailed technical introduction.

Throughout this paper, let $\mathbb{F}$ be a field with characteristic 0. All the vector spaces are assumed over $\mathbb{F}$. Denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set of nonnegative numbers $\{0, 1, 2, 3, ...\}$. We shall always identify $\mathbb{Z}$ with $\mathbb{Z}1_{\mathbb{F}}$ when the context is clear.

Let $\mathcal{A} = \mathbb{F}[t_1, t_2, ..., t_n]$ be the algebra of polynomials in n variables. A *derivation* ∂ of $\mathcal{A}$ is linear transformation of $\mathcal{A}$ such that

$$\partial(uv) = \partial(u)v + u\partial(v) \qquad \text{for } \; u, v \in \mathcal{A}. \tag{1.1}$$

Typical derivations are $\{\partial_{t_1}, \partial_{t_2}, ..., \partial_{t_n}\}$, the operators of taking partial derivatives. The space $\mathrm{Der}\,\mathcal{A}$ of all the derivations of $\mathcal{A}$ forms a Lie algebra. Identifying the elements of $\mathcal{A}$ with their corresponding multiplication operators, we have

$$\mathrm{Der}\,\mathcal{A} = \sum_{i=1}^{n} \mathcal{A}\partial_{t_i}, \tag{1.2}$$

which forms a simple Lie algebra. The Lie algebra $\mathrm{Der}\,\mathcal{A}$ is called a *Witt algebra of rank* n, usually denoted as $\mathcal{W}(n, \mathbb{F})$. The Lie algebra $\mathcal{W}(n, \mathbb{F})$ acts on the Grassmann algebra $\hat{\mathcal{A}}$ of differential forms over $\mathcal{A}$ as follows.

$$\partial(df) = d(\partial(f)), \;\; \partial(\omega \wedge \nu) = \partial(\omega) \wedge \nu + \omega \wedge \partial(\nu) \tag{1.3}$$

for $f \in \mathcal{A}, \; \omega, \nu \in \hat{\mathcal{A}}, \; \partial \in \mathcal{W}(n, \mathbb{F})$.

For $\partial = \sum_{i=1}^{n} f_i \partial_{t_i} \in \mathcal{W}(n, \mathbb{F})$, we define the *divergence of* ∂ by

$$\mathrm{div}\,\partial = \sum_{i=1}^{n} \partial_{t_i}(f_i). \tag{1.4}$$

Set

$$\mathcal{S}(n, \mathbb{F}) = \{\partial \in \mathcal{W}(n, \mathbb{F}) \mid \mathrm{div}\,\partial = 0\}. \tag{1.5}$$

Then $\mathcal{S}(n, \mathbb{F})$ is the divergence-free Lie subalgebra of $\mathcal{W}(n, \mathbb{F})$. Moreover, $\mathcal{S}(n, \mathbb{F})$ is the subalgebra of $\mathcal{W}(n, \mathbb{F})$ annihilating the volume form; that is,

$$\mathcal{S}(n, \mathbb{F}) = \{\partial \in \mathcal{W}(n, \mathbb{F}) \mid \partial(dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n) = 0\}. \tag{1.6}$$

When $\mathbb{F} = \mathbb{R}$ the field of real numbers, e^{∂} is a volume-preserving diffeomorphism for each $\partial \in \mathcal{S}(n, \mathbb{F})$. The algebra $\mathcal{S}(n, \mathbb{F})$ is called a *Lie algebra of Special type.*

For any positive integer n, an additive subgroup G of $\mathbb{F}^n$ is called *nondegenerate* if G contains an $\mathbb{F}$-basis of $\mathbb{F}^n$. Let $\ell_1, \; \ell_2$ and ℓ_3 be three nonnegative integers such that

$$\ell = \ell_1 + \ell_2 + \ell_3 > 0. \tag{1.7}$$

Take any nondegenerate additive subgroup Γ of $\mathbb{F}^{\ell_2+\ell_3}$ and $\Gamma = \{0\}$ when $\ell_2 + \ell_3 = 0$. Let $\mathcal{A}(\ell_1, \ell_2, \ell_3; \Gamma)$ be a free $\mathbb{F}[t_1, t_2, ..., t_{\ell_1+\ell_2}]$-module with the basis

$$\{x^{\alpha} \mid \alpha \in \Gamma\}. \tag{1.8}$$

Viewing $\mathcal{A}(\ell_1,\ell_2,\ell_3;\Gamma)$ as a vector space over $\mathbb{F}$, we define a commutative associative algebraic operation "$\cdot$" on $\mathcal{A}(\ell_1,\ell_2,\ell_3;\Gamma)$ by

$$(\zeta x^\alpha)\cdot(\eta x^\beta)=\zeta\eta x^{\alpha+\beta}\qquad \text{for }\ \zeta,\eta\in\mathbb{F}[t_1,t_2,...,t_{\ell_1+\ell_2}],\ \alpha,\beta\in\Gamma. \tag{1.9}$$

Note that x^0 is the identity element, which is denoted as 1 for convenience. When the context is clear, we shall omit the notation "$\cdot$" in any associative algebra product.

We define the linear transformations

$$\{\partial_{t_1},...,\partial_{t_{\ell_1+\ell_2}},\partial_1^*,...,\partial_{\ell_2+\ell_3}^*\} \tag{1.10}$$

on $\mathcal{A}(\ell_1,\ell_2,\ell_3;\Gamma)$ by

$$\partial_{t_i}(\zeta x^\alpha)=\partial_{t_i}(\zeta)x^\alpha,\ \ \partial_j^*(\zeta x^\alpha)=\alpha_j\zeta x^\alpha \tag{1.11}$$

for $\zeta\in\mathbb{F}[t_1,t_2,...,t_{\ell_1+\ell_2}]$ and $\alpha=(\alpha_1,...,\alpha_{\ell_2+\ell_3})\in\Gamma$. Then $\{\partial_{t_1},...,\partial_{t_{\ell_1+\ell_2}},\partial_1^*,...,\partial_{\ell_2+\ell_3}^*\}$ are mutually commutative derivations of $\mathcal{A}(\ell_1,\ell_2,\ell_3;\Gamma)$.

Throughout this paper, we shall use the following notation of index set

$$\overline{m,n}=\{m,m+1,m+2,...,n\}\qquad\text{for }\ m,n\in\mathbb{Z},\ m\leq n. \tag{1.12}$$

Moreover, we also take $\overline{m,n}=\emptyset$ if $m>n$. Set

$$\partial_i=\partial_{t_i},\ \ \partial_{\ell_1+j}=\partial_j^*+\partial_{t_{\ell_1+j}},\ \ \partial_{\ell_1+\ell_2+l}=\partial_{\ell_2+l}^* \tag{1.13}$$

for $i\in\overline{1,\ell_1}$, $j\in\overline{1,\ell_2}$ and $l\in\overline{1,\ell_3}$. Then $\{\partial_i\mid i\in\overline{1,\ell}\}$ is an $\mathbb{F}$-linearly independent set of derivations. Let

$$\mathcal{D}=\sum_{i=1}^{\ell}\mathbb{F}\partial_i \tag{1.14}$$

and

$$\mathcal{W}(\ell_1,\ell_2,\ell_3;\Gamma)=\mathcal{A}(\ell_1,\ell_2,\ell_3;\Gamma)\mathcal{D}. \tag{1.15}$$

Then $\mathcal{W}(\ell_1,\ell_2,\ell_3;\Gamma)$ is a simple Lie algebra of Witt type constructed by the second author in [X1].

A linear transformation T on a vector space V is called *locally-finite* if

$$\dim(\mathrm{Span}\,\{T^n(u)\mid n\in\mathbb{N}\})<\infty\qquad\text{for }\ u\in V. \tag{1.16}$$

Zhang and the authors of this paper proved that the pairs $(\mathcal{A}(\ell_1,\ell_2,\ell_3;\Gamma),\mathcal{D})$ for different parameters $(\ell_1,\ell_2,\ell_3;\Gamma)$ enumerate all the pairs $(\mathcal{A},\mathcal{D})$ of a commutative associative algebra $\mathcal{A}$ with an identity element and its finite-dimensional commutative locally-finite derivation subalgebra $\mathcal{D}$ such that the commutative associative algebra is derivation-simple with respect to $\mathcal{D}$ and

$$\bigcap_{d\in\mathcal{D}}\ker_d=\mathbb{F}. \tag{1.17}$$

Denote by $M_{m\times n}$ the algebra of $m\times n$ matrices with entries in $\mathbb{F}$ and by GL_m the group of invertible $m\times m$ matrices with entries in $\mathbb{F}$. Set

$$G_{\ell_2,\ell_3}=\left\{\left(\begin{array}{cc}A & 0_{\ell_2\times\ell_3}\\ B & C\end{array}\right)\mid A\in GL_{\ell_2},\ B\in M_{\ell_2\times\ell_3},\ C\in GL_{\ell_3}\right\}, \tag{1.18}$$

where $0_{\ell_2\times\ell_3}$ is the $\ell_2\times\ell_3$ matrix whose entries are zero. Then G_{ℓ_2,ℓ_3} forms a subgroup of $GL_{\ell_2+\ell_3}$. Define an action of G_{ℓ_2,ℓ_3} on $\mathbb{F}^{\ell_2+\ell_3}$ by

$$g(\alpha)=\alpha g^{-1}\ \text{(matrix multiplication)}\qquad\text{for }\ \alpha\in\mathbb{F}^{\ell_2+\ell_3},\ g\in G_{\ell_2,\ell_3}. \tag{1.19}$$

For any nondegenerate additive subgroup Υ of $\mathbb{F}^{\ell_2+\ell_3}$ and $g\in G_{\ell_2,\ell_2}$, the set

$$g(\Upsilon)=\{g(\alpha)\mid\alpha\in\Upsilon\} \tag{1.20}$$

also forms a nondegenerate additive subgroup of $\mathbb{F}^{\ell_2+\ell_3}$. Let

$$\Omega_{\ell_2+\ell_3}=\text{the set of nondegenerate additive subgroups of }\mathbb{F}^{\ell_2+\ell_3}. \tag{1.21}$$

We have an action of G_{ℓ_2,ℓ_3} on $\Omega_{\ell_2+\ell_3}$ by (1.20). Define the moduli space

$$\mathcal{M}^W_{\ell_2,\ell_3}=\Omega_{\ell_2+\ell_3}/G_{\ell_2,\ell_3}, \tag{1.22}$$

which is the set of G_{ℓ_2,ℓ_3}-orbits in $\Omega_{\ell_2+\ell_3}$.

Another related result in [SXZ] is as follows. The Lie algebras $\mathcal{W}(\ell_1,\ell_2,\ell_3;\Gamma)$ and $\mathcal{W}(\ell_1',\ell_2',\ell_3';\Gamma')$ are isomorphic if and only if $(\ell_1,\ell_2,\ell_3)=(\ell_1',\ell_2',\ell_3')$ and there exists an element $g\in G_{\ell_2,\ell_3}$ such that $g(\Gamma)=\Gamma'$. In particular, there exists a one-to-one correspondence between the set of isomorphism classes of the Lie algebras of the form (1.15) and the following set:

$$SW=\{(\ell_1,\ell_2,\ell_3,\varpi)\mid(0,0,0)\neq(\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3,\ \varpi\in\mathcal{M}^W_{\ell_2,\ell_3}\}. \tag{1.23}$$

In other words, the set SW is the structure space of the simple Lie algebras of Witt type in the form (1.15).

We define the divergence by

$$\mathrm{div}\,\partial=\sum_{i=1}^{\ell}\partial_i(u_i)\qquad\text{for }\ \partial=\sum_{i=1}^{\ell}u_i\partial_i\in\mathcal{W}(\ell_1,\ell_2,\ell_3;\Gamma), \tag{1.24}$$

and set

$$\mathcal{S}(\ell_1,\ell_2,\ell_3;\Gamma)=\{\partial\in\mathcal{W}(\ell_1,\ell_2,\ell_3;\Gamma)\mid\mathrm{div}\,\partial=0\}. \tag{1.25}$$

Let $\rho\in\Gamma$ be any element. Then the space

$$\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)=x^\rho\mathcal{S}(\ell_1,\ell_2,\ell_3;\Gamma) \tag{1.26}$$

forms a Lie subalgebra of the Lie algebra $\mathcal{W}(\ell_1,\ell_2,\ell_3;\Gamma)$, which is simple if $\ell_1+\ell_2>0$ or $\rho=0$ or $\ell\geq 3$ by the proofs of the simplicity of the Lie algebras of Special type in [DZ] and [X1]. When $\ell_1+\ell_2=0,\ \ell_3=2$ and $\rho\neq 0$, the derived subalgebra $(\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma))^{(1)}$ is simple and has codimension one in $\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)$. The Lie algebra $\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)$ was introduced by the second author in [X1], as a nongraded generalization of graded simple Lie algebras of Special type. We also call it a *divergence-free Lie algebra* in this paper. The special case $\mathcal{S}(\ell_1,0,\ell_3;\rho,\Gamma)$ was studied by Osborn [O], Djokovic and Zhao [DZ], and Zhao [Z].

Define the moduli space

$$\mathcal{M}^{\mathcal{S}}_{\ell_2,\ell_3}=(\Gamma\times\Omega_{\ell_2+\ell_3})/G_{\ell_2,\ell_3}, \tag{1.27}$$

where the action of G_{ℓ_2,ℓ_3} on $\Gamma\times\Omega_{\ell_2+\ell_3}$ is defined by $g(\rho,\Upsilon)=(g(\rho),g(\Upsilon))$ for $(\rho,\Upsilon)\in\Gamma\times\Omega_{\ell_2+\ell_3}$ (cf. (1.18)-(1.21)). The main theorem of this paper is as follows.

Main Theorem. *The Lie algebras $\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)$ and $\mathcal{S}(\ell_1',\ell_2',\ell_3';\rho',\Gamma')$ with $\ell\geq 3$ are isomorphic if and only if $(\ell_1,\ell_2,\ell_3)=(\ell_1',\ell_2',\ell_3')$ and there exists an element $g\in G_{\ell_2,\ell_3}$ such that $g(\Gamma)=\Gamma'$, and $g(\rho)=\rho'$ if $\ell_1=0$. In particular, there exists a one-to-one correspondence between the set of isomorphism classes of the Lie algebras of the form (1.26) and the set SW in (1.23) if $\ell_1>0$, and between the set of isomorphism classes of the Lie algebras of the form (1.26) and the following set*:

$$SS=\{(\ell_1,\ell_2,\ell_3,\varpi)\mid(0,0,0)\neq(\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3,\ \varpi\in\mathcal{M}^{\mathcal{S}}_{\ell_2,\ell_3}\} \tag{1.28}$$

if $\ell_1=0$.

When $\ell=2$, the Lie algebra $\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)$ is also a Lie algebra of Hamiltonian type. We shall determine the structure of the Lie algebras of Hamiltonian type in a subsequent paper.

The paper is organized as follows. In Section 2, we shall present some basic properties of the divergence-free Lie algebras. Structure of derivation algebras of the divergence-free Lie algebras will be determined in Section 3. Using the results in the above two sections, we shall give the proof of the main theorem in Section 4.

2 Some Basic Properties of the Lie algebras

In this section, we shall present some basic properties of the divergence-free Lie algebra $\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)$. The related assumptions and settings are the same as in the paragraphs of (1.7)-(1.15) and (1.24)-(1.26).

For convenience, we denote

$$\mathcal{D}_1=\sum_{i=1}^{\ell_1}\mathbb{F}\partial_i,\;\;\mathcal{D}_2=\sum_{j=1}^{\ell_2}\mathbb{F}\partial_{\ell_1+j},\;\;\mathcal{D}_3=\sum_{p=1}^{\ell_3}\mathbb{F}\partial_{\ell_1+\ell_2+p} \tag{2.1}$$

(cf. (1.13)) and

$$\mathcal{A}=\mathcal{A}(\ell_1,\ell_2,\ell_3;\Gamma),\;\;\mathcal{W}=\mathcal{W}(\ell_1,\ell_2,\ell_3;\Gamma),\;\;\mathcal{S}=\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma) \tag{2.2}$$

(cf. (1.9), (1.15), (1.26)). Then

$$\mathcal{D}=\mathcal{D}_1\oplus\mathcal{D}_2\oplus\mathcal{D}_3 \tag{2.3}$$

(cf. (1.14)). Moreover, we define

$$D_{p,q}(u)=x^{\rho}(\partial_q(x^{-\rho}u)\partial_p-\partial_p(x^{-\rho}u)\partial_q)\qquad\text{for }\;p,q\in\overline{1,\ell},\;u\in\mathcal{A}. \tag{2.4}$$

It can be proved that

$$\mathcal{S}=\text{Span}\,\{D_{p,q}(u)\mid p,q\in\overline{1,\ell};\;u\in\mathcal{A}\}. \tag{2.5}$$

Denote

$$t^{\vec{i}}=t_1^{i_1}t_2^{i_2}\cdots t_{\ell_1+\ell_2}^{i_{\ell_1+\ell_2}},\;\;x^{\alpha,\vec{i}}=t^{\vec{i}}x^{\alpha}\qquad\text{for }\;\vec{i}=(i_1,...,i_{\ell_1+\ell_2})\in\mathbb{N}^{\ell_1+\ell_2},\;\alpha\in\Gamma \tag{2.6}$$

and

$$a_{[i]}=(0,...,0,\overset{i}{a},0,...,0)\qquad\text{for }\;a\in\mathbb{F}. \tag{2.7}$$

In the rest of this paper, we shall use the convention that if a notation has not been defined but technically appears in an expression, we treat it as zero. For instance, when we use the notations $\alpha\in\Gamma$ and $\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}$, we teat $\alpha_p=0$ and $i_{\ell_1+\ell_2+q}=0$ for $p\leq 0$ and $q>0$ if they appear in an expression. Note

$$\begin{aligned}D_{p,q}(x^{\alpha,\vec{i}})\;&=\;x^{\alpha,\vec{i}}(((\alpha_{q-\ell_1}-\rho_{q-\ell_1})\partial_p-(\alpha_{p-\ell_1}-\rho_{p-\ell_1})\partial_q)\\&\quad+i_qx^{\alpha,\vec{i}-1_{[q]}}\partial_p-i_px^{\alpha,\vec{i}-1_{[p]}}\partial_q\end{aligned} \tag{2.8}$$

for $p,q\in\overline{1,\ell}$, $\alpha\in\Gamma$ and $\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}$. Set

$$\mathcal{S}_{\alpha}=\text{Span}\,\{D_{p,q}(x^{\alpha,\vec{i}})\mid p,q\in\overline{1,\ell};\;\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}\}\qquad\text{for }\;\alpha\in\Gamma. \tag{2.9}$$

Then

$$\mathcal{S}=\bigoplus_{\alpha\in\Gamma}\mathcal{S}_{\alpha}, \tag{2.10}$$

is a Γ-graded Lie algebra, and

$$\mathcal{S}_0=\text{Span}\,\{t^{\vec{i}}(-\rho_{q-\ell_1}\partial_p+\rho_{p-\ell_1}\partial_q)+i_qt^{\vec{i}-1_{[q]}}\partial_p-i_pt^{\vec{i}-1_{[p]}}\partial_q\mid p,q\in\overline{1,\ell},\;\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}\} \tag{2.11}$$

is a Lie subalgebra of $\mathcal{S}$.

We define a bilinear product from $\mathcal{D}\times\Gamma\to\mathbb{F}$ by

$$\alpha(\partial)=\langle\partial,\alpha\rangle=\sum_{i=1}^{\ell_2+\ell_3}a_{\ell_1+i}\alpha_i\qquad\text{for }\ \partial=\sum_{i=1}^{\ell}a_i\partial_i\in\mathcal{D},\ \alpha=(\alpha_1,...,\alpha_{\ell_2+\ell_3})\in\Gamma.\tag{2.12}$$

For any $\alpha\in\Gamma$, let

$$\mathcal{D}_\alpha=\begin{cases}\{0\} & \text{if } \ell_1=\ell_2=0,\alpha=0,\\ \mathcal{D}_3 & \text{if } \ell_1+\ell_2=1,\alpha=0,\\ \{\partial\in\mathcal{D}\mid\langle\partial,\alpha\rangle=0\} & \text{otherwise.}\end{cases}\tag{2.13}$$

In particular, we have

$$\dim\mathcal{D}_\alpha\geq\ell-1\geq 2\ \text{ if }\ \ell_1+\ell_2\geq 1\ \text{ or }\ \alpha\neq 0.\tag{2.14}$$

Denote

$$|\vec{i}|=\sum_{p=1}^{\ell_1+\ell_2}i_p\qquad\text{for }\ \vec{i}\in\mathbb{N}^{\ell_1+\ell_2}.\tag{2.15}$$

Define a total order on $\mathbb{N}^{\ell_1+\ell_2}$ by:

$$\vec{i}<\vec{j}\ \text{ if }\ |\vec{i}|<|\vec{j}|\ \text{ or }\ |\vec{i}|=|\vec{j}|\ \text{ with }\ i_1=j_1,...,i_{p-1}=j_{p-1},i_p<j_p\tag{2.16}$$

for some $p\in\overline{1,\ell_1+\ell_2}$. For any

$$u=x^\alpha(t^{\vec{i}}\partial_{\vec{i}}+\sum_{\vec{j}<\vec{i}}t^{\vec{j}}\partial_{\vec{j}})\in\mathcal{S}_\alpha\ \text{ with }\ \partial_{\vec{i}},\partial_{\vec{j}}\in\mathcal{D}\ \text{ such that }\ \partial_{\vec{i}}\neq 0,\tag{2.17}$$

we define

$$u_{ld}=x^{\alpha,\vec{i}}\partial_{\vec{i}}\tag{2.18}$$

to be the *leading term* of u. In this case, we say that u has *leading level* $|\vec{i}|$ and that u has *leading degree* $\vec{i}$, denoted by $d(u_{ld})=\vec{i}$.

Set

$$\mathcal{S}_\alpha^{[\vec{i}]}=\{u\in\mathcal{S}_\alpha\mid d(u_{ld})\leq\vec{i}\},\quad\mathcal{S}_\alpha^{(\vec{i})}=\{u\in\mathcal{S}_\alpha\mid d(u_{ld})<\vec{i}\}.\tag{2.19}$$

and

$$\mathcal{S}^{[\vec{i}]}=\bigoplus_{\alpha\in\Gamma}\mathcal{S}_\alpha^{[\vec{i}]},\quad\mathcal{S}^{(\vec{i})}=\bigoplus_{\alpha\in\Gamma}\mathcal{S}_\alpha^{(\vec{i})}.\tag{2.20}$$

Then

$$\mathcal{S}^{[0]}=\bigoplus_{\alpha\in\Gamma}\mathcal{S}_\alpha^{[0]}\tag{2.21}$$

is a Lie subalgebra of $\mathcal{S}$.

Lemma 2.1. *For $\alpha\in\Gamma$, we have*

$$\mathcal{S}_\alpha^{[0]}=\{x^\alpha\partial\mid\partial\in\mathcal{D}_{\alpha-\rho}\}.\tag{2.22}$$

Proof. First we assume $\alpha \in \Gamma\backslash\{\rho\}$. There exists $q \in \overline{\ell_1+1,\ell}$ such that $\alpha_{q-\ell_1} \neq \rho_{q-\ell_1}$. For $\partial = \sum_{p=1}^{\ell} a_p\partial_p \in \mathcal{D}_{\alpha-\rho}$, we have

$$\begin{aligned}\sum_{p=1}^{\ell} a_p D_{p,q}(x^\alpha) &= x^\alpha(\sum_{p=1}^{\ell} a_p(\alpha_{q-\ell_1}-\rho_{q-\ell_1})\partial_p + \sum_{p=1}^{\ell} a_p(\alpha_{p-\ell_1}-\rho_{p-\ell_1})\partial_q) \\ &= x^\alpha((\alpha_{q-\ell_1}-\rho_{q-\ell_1})\partial + \langle\partial,\alpha-\rho\rangle\partial_q) \\ &= (\alpha_{q-\ell_1}-\rho_{q-\ell_1})x^\alpha\partial \end{aligned} \tag{2.23}$$

by (2.13). Thus $x^\alpha\partial \in \mathcal{S}_\alpha^{[0]}$. Conversely, (2.8) and (2.9) imply

$$\begin{aligned}\mathcal{S}_\alpha^{[0]} = &\ \text{Span}\,\{D_{p,q}(x^{\alpha,\vec{i}}) \mid \vec{i}=\vec{0} \text{ or } \alpha_{p-\ell_1}-\rho_{p-\ell_1}=0, \\ &\ \alpha_{q-\ell_1}-\rho_{q-\ell_1}=0 \text{ and } \vec{i}=1_{[p]} \text{ or } 1_{[q]}\}, \end{aligned} \tag{2.24}$$

and we see that $\mathcal{S}_\alpha^{[0]} \subseteq \{x^\alpha\partial \mid \partial\in\mathcal{D}_{\alpha-\rho}\}$.

Next we assume $\alpha=\rho$. If $\ell_1+\ell_2=0$, then we have $\mathcal{S}_\rho^{[0]}=\{0\}$ by (2.8) and (2.24). Suppose $\ell_1+\ell_2\geq 1$. Let $p\in\overline{1,\ell}$. If $\ell_1+\ell_2\geq 2$ or $p\in\overline{\ell_1+\ell_2+1,\ell}$, then we can choose $q\in\overline{1,\ell_1+\ell_2}\backslash\{p\}$ and get

$$x^\rho\partial_p = D_{p,q}(x^{\rho,1_{[q]}}) \in \mathcal{S}_\rho^{[0]}. \tag{2.25}$$

However, if $\ell_1+\ell_2=1$ and $p=1$, then (2.8) and (2.24) imply $x^\rho\partial_1\notin\mathcal{S}_\rho^{[0]}$. So we obtain (2.22) by (2.13). $\square$

We define a linear function $\chi_p:\mathcal{D}\to\mathbb{F}$ by

$$\chi_p(\partial)=a_p \qquad \text{for } \partial=\sum_{p=1}^{\ell} a_p\partial\in\mathcal{D}. \tag{2.26}$$

Lemma 2.2. *For $\alpha\in\Gamma$, $\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}$ and $\partial=\sum_{p=1}^{\ell} a_p\partial_p\in\mathcal{D}\backslash\{0\}$, there exists an element in $\mathcal{S}_\alpha$ with leading term $x^{\alpha,\vec{i}}\partial$ if and only if $\partial\in\mathcal{D}_{\alpha-\rho}$, and $i_{\ell_1+\ell_2}=0$ or $a_{\ell_1+\ell_2}=0$ if $\alpha=\rho$.*

Proof. Suppose $\alpha_{q-\ell_1}\neq\rho_{q-\ell_1}$ for some $q\in\overline{\ell_1+1,\ell}$. Let $\partial\in\mathcal{D}_{\alpha-\rho}$. As in (2.23), the element

$$\begin{aligned}&(\alpha_{q-\ell_1}-\rho_{q-\ell_1})^{-1}\sum_{p=1}^{\ell} a_p\partial_{p,q}(x^{\alpha,\vec{i}}) \\ =\ & x^{\alpha,\vec{i}}\partial + (\alpha_{q-\ell_1}-\rho_{q-\ell_1})^{-1}\sum_{p=1}^{\ell} a_p(i_q x^{\alpha,\vec{i}-1_{[q]}}\partial_p - i_p x^{\alpha,\vec{i}-1_{[p]}}\partial_q)\end{aligned} \tag{2.27}$$

is in $\mathcal{S}_\alpha$ with the leading term $x^{\alpha,\vec{i}}\partial$. Conversely, if $\mathcal{S}_\alpha$ has an element with the leading term $x^{\alpha,\vec{i}}\partial$, then (2.8) and (2.9) imply $\partial\in\mathcal{D}_{\alpha-\rho}$.

Assume $\alpha=\rho$. Let $\partial=\sum_{p=1}^{\ell}a_p\partial_p\in\mathcal{D}_0$. If $a_{\ell_1+\ell_2}=0$, then

$$
\begin{aligned}
&(i_{\ell_1+\ell_2}+1)^{-1}\sum_{p=1}^{\ell}a_pD_{p,\ell_1+\ell_2}(x^{\rho,\vec{i}+1_{[\ell_1+\ell_2]}})\\
=\ &x^{\rho,\vec{i}}\partial-(i_{\ell_1+\ell_2}+1)^{-1}\sum_{p=1}^{\ell_1+\ell_2-1}a_pi_px^{\rho,\vec{i}-1_{[p]}+1_{[\ell_1+\ell_2]}}\partial_{\ell_1+\ell_2}
\end{aligned}
\tag{2.28}
$$

is an element in $\mathcal{S}_\rho$ with the leading term $x^{\rho,\vec{i}}\partial$. If $a_{\ell_1+\ell_2}\neq 0$, then $i_{\ell_1+\ell_2}=0$ by our assumption. By (2.13), we must have $\ell_1+\ell_2\geq 2$. So we can write $\partial=a_{\ell_1+\ell_2}\partial_{\ell_1+\ell_2}+\partial'$ such that $\partial'\in\mathcal{D}_0$ and $\chi_{\ell_1+\ell_2}(\partial')=0$. Then (2.28) shows that there exists an element in $\mathcal{S}_\rho$ with the leading term $x^{\rho,\vec{i}}\partial'$. Moreover, we have

$$
x^{\rho,\vec{i}}\partial_{\ell_1+\ell_2}=(i_1+1)^{-1}D_{\ell_1+\ell_2,1}(x^{\rho,\vec{i}+1_{[1]}})\in\mathcal{S}_\rho. \tag{2.29}
$$

Thus there exists an element in $\mathcal{S}_\rho$ with leading term $x^{\rho,\vec{i}}\partial$. Similarly, if an element of $\mathcal{S}_\rho$ has a leading term $x^{\rho,\vec{i}}\partial$, then we have $\partial\in\mathcal{D}_0$ and $\chi_{\ell_1+\ell_2}(\partial)=0$ or $i_{\ell_1+\ell_2}=0$ by (2.8) and (2.9). $\square$

Lemma 2.3. *If* $\ell_1\geq 1$, then $\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)\cong\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho',\Gamma)$ *for any* $\rho,\rho'\in\Gamma$.

Proof. For any $\alpha^{(1)},...,\alpha^{(\ell_1)}\in\Gamma$, we define an automorphism ψ of the associate algebra $\mathcal{A}$ (cf. (2.2)) by

$$
\psi(x^{\alpha,\vec{i}})=x^{\alpha+\sum_{p=1}^{\ell_1}i_p\alpha^{(p)},\vec{i}}\qquad\text{for }\ (\alpha,\vec{i})\in\Gamma\times\mathbb{N}^{\ell_1+\ell_2}. \tag{2.30}
$$

This induces a Lie algebra automorphism $\overline{\psi}$ on $\mathcal{W}$ (cf. (2.2)):

$$
\overline{\psi}(x^{\alpha,\vec{i}}\partial_q)=x^{\alpha+\sum_{p=1}^{\ell_1}i_p\alpha^{(p)}-\alpha^{(q)},\vec{i}}\partial_q\qquad\text{for }\ (\alpha,\vec{i})\in\Gamma\times\mathbb{N}^{\ell_1+\ell_2},\ q\in\overline{1,\ell_1}; \tag{2.31}
$$

$$
\overline{\psi}(x^{\alpha,\vec{i}}\partial_q)=x^{\alpha+\sum_{p=1}^{\ell_1}i_p\alpha^{(p)},\vec{i}}(\partial_q-\sum_{p=1}^{\ell_1}t_p\partial_p)\qquad\text{for }\ (\alpha,\vec{i})\in\Gamma\times\mathbb{N}^{\ell_1+\ell_2},\ q\in\overline{\ell_1+1,\ell}. \tag{2.32}
$$

It can be verified by (1.24)-(1.26) that the automorphism $\overline{\psi}$ of $\mathcal{W}$ induces an isomorphism

$$
\overline{\psi}:\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)\to\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho^*,\Gamma)\ \text{ with }\ \rho^*=\rho+\sum_{p=1}^{\ell_1}\alpha^{(p)}. \tag{2.33}
$$

Taking $\alpha^{(1)}=\rho'-\rho$ and $\alpha^{(2)}=\cdots=\alpha^{(\ell_1)}=0$, we have $\rho^*=\rho'$. $\square$

For convenience, we shall always take

$$
\rho=0\qquad\text{if }\ \ell_1\geq 1 \tag{2.34}
$$

in the rest of this paper.

3 Structure of the Derivation Algebras

In this section, we shall determine the structure of the derivation algebra of the divergence-free Lie algebra $\mathcal{S}(\ell_1,\ell_2,\ell_3;\rho,\Gamma)$ defined by (1.26), which has been simply denoted as $\mathcal{S}$.

In the rest of this paper,

$$\text{we choose an element } \ \bar{x}^{\alpha,\vec{i}}\partial\in\mathcal{S} \ \text{ with the leading term } \ x^{\alpha,\vec{i}}\partial \tag{3.1}$$

for each

$$(\alpha,\vec{i})\in\Gamma\times\mathbb{N}^{\ell_1+\ell_2},\ \partial\in\mathcal{D}_{\alpha-\rho} \text{ such that } i_{\ell_1+\ell_2}=0 \text{ or } \chi_{\ell_1+\ell_2}(\partial)=0 \text{ if } \alpha=\rho. \tag{3.2}$$

Recall that a derivation d of the Lie algebra $\mathcal{S}$ is a linear transformation on $\mathcal{S}$ such that

$$d([u_1,u_2])=[d(u_1),u_2]+[u_1,d(u_2)] \qquad \text{for } \ u_1,u_2\in\mathcal{S}. \tag{3.3}$$

Denote

$$\mathrm{Der}\,\mathcal{S}=\text{the space of the derivations of } \mathcal{S}. \tag{3.4}$$

It is well known that $\mathrm{Der}\,\mathcal{S}$ forms a Lie algebra with respect to the commutator of linear transformations of $\mathcal{S}$, and $\mathrm{ad}_{\mathcal{S}}$ is an ideal of $\mathrm{Der}\,\mathcal{S}$.

Recall that $\mathcal{S}$ is a Γ-graded Lie algebra with the grading defined by (2.8) and (2.9). For $\alpha\in\Gamma$, we set

$$(\mathrm{Der}\,\mathcal{S})_\alpha=\{d\in\mathrm{Der}\,\mathcal{S}\mid d(\mathcal{S}_\beta)\subset\mathcal{S}_{\alpha+\beta} \text{ for } \beta\in\Gamma\}. \tag{3.5}$$

Fix an element $\alpha\in\Gamma$. Recall the notations in (2.20). Consider a nonzero element

$$d\in(\mathrm{Der}\,\mathcal{S})_\alpha \ \text{ such that } \ d(\mathcal{S}^{[\vec{j}]})\subset\mathcal{S}^{[\vec{i}+\vec{j}]} \ \text{ for any } \ \vec{j}\in\mathbb{N}^{\ell_1+\ell_2}, \tag{3.6}$$

where $\vec{i}\in\mathbb{Z}^{\ell_1+\ell_2}$ is a fixed element. Define the order in $\mathbb{Z}^{\ell_1+\ell_2}$ as in (2.16). We remark that if $\vec{j}\in\mathbb{Z}^{\ell_1+\ell_2}\backslash\mathbb{N}^{\ell_1+\ell_2}$, it does not mean that $\mathcal{S}^{[\vec{j}]}=0$ since it is possible that the level $|\vec{j}|>0$. But if $\vec{j}\in\mathbb{Z}^{\ell_1+\ell_2}\backslash\mathbb{N}^{\ell_1+\ell_2}$, then we have $\mathcal{S}^{[\vec{j}]}=\mathcal{S}^{(\vec{j})}$ (cf. (2.20)). Moreover, we can assume

$$\vec{i} \ \text{ in (3.6) is the minimal element satisfying the condition.} \tag{3.7}$$

For any $(\beta,\vec{j})\in\Gamma\times\mathbb{N}^{\ell_1+\ell_2}$, we define a linear map $e_{\beta,\vec{j}}:\mathcal{D}_{\beta-\rho}\to\mathcal{D}_{\alpha+\beta-\rho}$ by

$$d(\bar{x}^{\beta,\vec{j}}\partial)\equiv\bar{x}^{\alpha+\beta,\vec{i}+\vec{j}}e_{\beta,\vec{j}}(\partial) \ \ (\mathrm{mod}\,\mathcal{S}^{(\vec{i}+\vec{j})}). \tag{3.8}$$

Using the assumption (3.2) and the notation in (3.1), we have

$$[\bar{x}^{\beta,\vec{j}}\partial',\bar{x}^{\gamma,\vec{k}}\partial'']\equiv\bar{x}^{\beta+\gamma,\vec{j}+\vec{k}}(\gamma(\partial')\partial''-\beta(\partial'')\partial') \ (\mathrm{mod}\,\mathcal{S}^{(\vec{j}+\vec{k})}). \tag{3.9}$$

Applying d to (3.9), we get

$$\begin{aligned} & \langle e_{\beta,\vec{j}}\partial',\gamma\rangle\partial''-\langle\partial'',\alpha+\beta\rangle e_{\beta,\vec{j}}\partial'+\langle\partial',\alpha+\gamma\rangle e_{\gamma,\vec{k}}\partial''-\langle e_{\gamma,\vec{k}}\partial'',\beta\rangle\partial' \\ = \ & e_{\beta+\gamma,\vec{j}+\vec{k}}(\langle\partial',\gamma\rangle\partial''-\langle\partial'',\beta\rangle\partial'). \end{aligned} \tag{3.10}$$

Lemma 3.1. *If $\mathcal{D}_\rho\not\subset\mathcal{D}_\alpha$ and $\alpha\neq 0$, then the derivation d in (3.6) is an inner derivation of $\mathcal{S}$.*

Proof. By assumption, there exists $\hat{\partial}\in\mathcal{D}_\rho\backslash\mathcal{D}_\alpha$ with $\langle\hat{\partial},\alpha\rangle=1$. Fix such a $\hat{\partial}$ and set

$$\tilde{\partial}=-e_{0,\vec{0}}\hat{\partial}. \tag{3.11}$$

Taking $\gamma=0$, $\vec{k}=\vec{0}$ and $\partial''=\hat{\partial}\in\mathcal{D}_{\gamma-\rho}$ in (3.10), we obtain

$$-\langle\hat{\partial},\alpha+\beta\rangle e_{\beta,\vec{j}}\partial'+\langle\partial',\alpha\rangle e_{0,\vec{0}}\hat{\partial}-\langle e_{0,\vec{0}}\hat{\partial},\beta\rangle\partial'=-\langle\hat{\partial},\beta\rangle e_{\beta,\vec{j}}\partial'. \tag{3.12}$$

By (3.11), (3.12) becomes

$$e_{\beta,\vec{j}}\partial'=\langle\tilde{\partial},\beta\rangle\partial'-\langle\partial',\alpha\rangle\tilde{\partial}\qquad\text{for any }\ (\beta,\vec{j},\partial')\ \text{ satisfying (3.2).} \tag{3.13}$$

Thus $e_{\beta,\vec{j}}$ does not depend on $\vec{j}$ for any β. Letting $\vec{j}=\vec{0}$ in (3.8), we have $\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}$ by (3.7). Moreover, we have $\alpha\neq\rho$ by the assumption $\mathcal{D}_\rho\not\subset\mathcal{D}_\alpha$. Letting $\beta=2\rho$ and $\partial'=\hat{\partial}\in\mathcal{D}_{\beta-\rho}$ in (3.13) and taking the bilinear product (2.12) of (3.13) with $\beta+\alpha-\rho$, we obtain $\langle\tilde{\partial},\alpha-\rho\rangle=0$ because $\langle e_{\beta,\vec{j}}\hat{\partial},\beta+\alpha-\rho\rangle=0$ by (3.8). Set

$$d'=d-\mathrm{ad}_{\bar{x}^{\alpha,\vec{i}}\tilde{\partial}}. \tag{3.14}$$

By (3.8) and (3.13), we obtain that

$$d'(\bar{x}^{\beta,\vec{j}}\partial')\in\mathcal{S}^{(\vec{i}+\vec{j})}\qquad\text{for any }\ (\beta,\vec{j},\partial')\ \text{ satisfying (3.2).} \tag{3.15}$$

By induction on $\vec{i}$ in (3.7), d' is an inner derivation. Hence d is an inner derivation. $\square$

Set

$$\mathcal{W}_\rho^{[0]}=x^\rho\mathcal{D} \tag{3.16}$$

(cf. (1.8), (1.14)).

Lemma 3.2. *Let d, $\vec{i}$ be the same as in (3.6) and (3.7) with $\alpha\neq 0$. If $\mathcal{D}_\rho\subseteq\mathcal{D}_\alpha$, then $\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}$ and d is an inner derivation when $\alpha\neq\rho$, or $d\in\mathrm{ad}_{\mathcal{W}_\rho^{[0]}}|_{\mathcal{S}}+\mathrm{ad}_{\mathcal{S}}$ when $\alpha=\rho$.*

Proof. Notice that any element of $\mathrm{ad}_{\mathcal{W}_\rho^{[0]}}|_{\mathcal{S}}$ are homogeneous derivations of $\mathcal{S}$ of degree ρ. By (2.13), the condition $\mathcal{D}_\rho\subseteq\mathcal{D}_\alpha$ and the assumption (2.34) imply $\ell_1=0$. So we have $\rho=a\alpha$ for some $a\in\mathbb{F}$. Note that in this case $\ell_2+\ell_3=\ell\geq 3$.

Claim 1. For any $\beta \in \Gamma \backslash \mathbb{F}\alpha$ and $\vec{j} \in \mathbb{N}^{\ell_1+\ell_2}$, there exists $a_{\beta,\vec{j}} \in \mathbb{F}$ such that

$$e_{\beta,\vec{j}}\partial' = a_{\beta,\vec{j}}\partial' \qquad \text{for } \partial' \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta. \tag{3.17}$$

Let β and ∂' be the same as in the above. Taking $\gamma = -\beta + \rho$ and $\partial'' \in \mathcal{D}_{\gamma-\rho} = \mathcal{D}_\beta$ in (3.10), we have that the third term of the left-hand side vanishes due to $\partial' \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta$, and the right-hand side also vanishes because of $\langle \partial', \gamma \rangle = \langle \partial'', \beta \rangle = 0$. Thus (3.10) becomes

$$\langle e_{\beta,\vec{j}}\partial', -\beta+\rho\rangle\partial'' - \langle \partial'', \alpha\rangle e_{\beta,\vec{j}}\partial' - \langle e_{-\beta+\rho,\vec{k}}\partial'', \beta\rangle\partial' = 0 \tag{3.18}$$

for $\partial' \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta$ and $\partial'' \in \mathcal{D}_\beta$. Taking the product (2.12) of (3.18) with $-\beta+\rho$, we obtain

$$\langle e_{\beta,\vec{j}}\partial', -\beta+\rho\rangle\langle \partial'', \rho-\alpha\rangle = 0 \tag{3.19}$$

by the facts that $\partial' \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta$ and $\partial'' \in \mathcal{D}_\beta$. If $\alpha \neq \rho$, then we take $\partial'' \in \mathcal{D}_\beta \backslash \mathcal{D}_\alpha$, and (3.19) forces $\langle e_{\beta,\vec{j}}\partial', -\beta+\rho\rangle = 0$. So (3.18) implies (3.17) with

$$a_{\beta,\vec{j}} = -\langle \partial'', \alpha\rangle^{-1}\langle e_{-\beta+\rho,0}, \partial''\rangle. \tag{3.20}$$

Suppose $\alpha = \rho$. Then $\rho \neq 0$ by our assumption $\alpha \neq 0$. We want to prove that the coefficient of the first term of (3.18) is zero. Taking $\gamma = -\rho$ and $\partial'' = \partial' \in \mathcal{D}_\rho \cap \mathcal{D}_\beta \backslash \{0\}$ in (3.10), we have that the second term and the third term of the left-hand side vanish, and so does the right-hand side. Thus (3.10) is equivalent to

$$\langle e_{\beta,\vec{j}}\partial', -\rho\rangle = \langle e_{-\rho,\vec{k}}\partial', \beta\rangle \qquad \text{for } \partial' \in \mathcal{D}_\rho \cap \mathcal{D}_\beta. \tag{3.21}$$

Next we take $\gamma = -\rho$, $\partial'' \in \mathcal{D}_\rho \cap \mathcal{D}_\beta \backslash \{0\}$ and $\partial' \in \mathcal{D}_{\beta-\rho} \backslash \mathcal{D}_\rho$. Then (3.10) becomes

$$\langle e_{\beta,\vec{j}}\partial', -\rho\rangle\partial'' - \langle e_{-\rho,\vec{k}}\partial'', \beta\rangle\partial' = \langle \partial', -\rho\rangle e_{\beta-\rho,\vec{j}+\vec{k}}\partial''. \tag{3.22}$$

Taking the product (2.12) of (3.22) with β, we get

$$-\langle e_{-\rho,\vec{k}}\partial'', \beta\rangle\langle \partial', \beta\rangle = \langle \partial', -\rho\rangle\langle e_{\beta-\rho,\vec{j}+\vec{k}}\partial'', \beta\rangle. \tag{3.23}$$

Since $\langle \partial', \beta\rangle = \langle \partial', \rho\rangle \neq 0$ by the fact $\partial' \in \mathcal{D}_{\beta-\rho} \backslash \mathcal{D}_\rho$, the above equation implies

$$\langle e_{-\rho,\vec{k}}\partial'', \beta\rangle = \langle e_{\beta-\rho,\vec{j}+\vec{k}}\partial'', \beta\rangle \qquad \text{for } \partial'' \in \mathcal{D}_\rho \cap \mathcal{D}_\beta. \tag{3.24}$$

Replacing β by $\beta+\rho$ in (3.24), we obtain

$$\langle e_{-\rho,\vec{k}}\partial'', \beta+\rho\rangle = \langle e_{\beta,\vec{j}+\vec{k}}\partial'', \beta+\rho\rangle \qquad \text{for } \partial'' \in \mathcal{D}_\rho \cap \mathcal{D}_\beta. \tag{3.25}$$

On the other hand, taking the product (2.12) of (3.18) with β and choosing $\partial'' \in \mathcal{D}_\beta \backslash \mathcal{D}_\rho$, we have

$$\langle e_{\beta,\vec{j}}\partial', \beta\rangle = 0 \qquad \text{for } \partial' \in \mathcal{D}_\beta \cap \mathcal{D}_\rho. \tag{3.26}$$

Taking $\beta = \gamma = -\rho$ and $\partial', \partial'' \in \mathcal{D}_\rho$ in (3.10), we have

$$\langle e_{-\rho,\vec{j}}\partial', -\rho\rangle\partial'' - \langle e_{-\rho,\vec{k}}\partial'', -\rho\rangle\partial' = 0. \tag{3.27}$$

By (2.14), $\dim \mathcal{D}_\rho \geq 2$. Thus (3.27) forces

$$\langle e_{-\rho,\vec{k}}\partial'', \rho\rangle = 0 \qquad \text{for } \partial'' \in \mathcal{D}_\rho. \tag{3.28}$$

Substituting (3.26) and (3.28) into (3.25), we obtain

$$\langle e_{-\rho,\vec{k}}\partial'', \beta\rangle = \langle e_{\beta,\vec{j}+\vec{k}}\partial'', \rho\rangle \qquad \text{for } \partial'' \in \mathcal{D}_\rho \cap \mathcal{D}_\beta. \tag{3.29}$$

Comparing this with (3.21), we have $\langle e_{\beta,\vec{j}}\partial', \rho\rangle = 0$, which together with (3.26) implies that the first term of (3.18) is zero. This completes the proof of (3.17).

Claim 2. There exists $\tilde{\partial}$ such that

$$e_{\beta,\vec{j}}\partial' = \langle\tilde{\partial}, \beta\rangle\partial' - \langle\partial', \alpha\rangle\tilde{\partial} \qquad \text{for } \partial' \in \mathcal{D}_{\beta-\rho},\ \beta \in \Gamma. \tag{3.30}$$

Given an element $\beta \in \Gamma\backslash\mathbb{F}\alpha$, we can take $\hat{\partial}_\beta \in \mathcal{D}_{\beta-\rho}$ such that $\langle\hat{\partial}_\beta, \alpha\rangle = 1$. So we have a vector space decomposition:

$$\mathcal{D}_{\beta-\rho} = \mathbb{F}\hat{\partial}_\beta \oplus (\mathcal{D}_\alpha \cap \mathcal{D}_\beta). \tag{3.31}$$

Moreover, we define

$$\partial_{\beta,\vec{j}} = a_{\beta,\vec{j}}\hat{\partial}_\beta - e_{\beta,\vec{j}}\hat{\partial}_\beta. \tag{3.32}$$

Then for any $\partial' = \partial'' + b\hat{\partial}_\beta \in \mathcal{D}_{\beta-\rho}$ with $\partial'' \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta$, we have

$$e_{\beta,\vec{j}}\partial' = a_{\beta,\vec{j}}\partial'' + b(a_{\beta,\vec{j}}\hat{\partial}_\beta - \partial_{\beta,\vec{j}}) = a_{\beta,\vec{j}}\partial' - \langle\partial', \alpha\rangle\partial_{\beta,\vec{j}} \tag{3.33}$$

for $\partial' \in \mathcal{D}_{\beta-\rho}$ and $\beta \in \Gamma\backslash\mathbb{F}\alpha$ by (3.17). Using this in (3.10), we obtain

$$\begin{aligned}
&(\langle\partial', \gamma\rangle(a_\beta + a_\gamma - a_{\beta+\gamma}) + \langle\partial', \alpha\rangle(a_\gamma - \langle\partial_\beta, \gamma\rangle))\partial'' - (\langle\partial'', \beta\rangle(a_\beta + a_\gamma - a_{\beta+\gamma}) \\
&+\langle\partial'', \alpha\rangle(a_\beta - \langle\partial_\gamma, \beta\rangle))\partial' + \langle\partial', \alpha\rangle\langle\partial'', \alpha\rangle(\partial_\beta - \partial_\gamma) + \langle\partial'', \beta\rangle\langle\partial', \alpha\rangle(\partial_\beta - \partial_{\beta+\gamma}) \\
&+\langle\partial', \gamma\rangle\langle\partial'', \alpha\rangle(\partial_{\beta+\gamma} - \partial_\gamma) = 0
\end{aligned} \tag{3.34}$$

for

$$\beta, \gamma, \beta+\gamma \in \Gamma\backslash\mathbb{F}\alpha,\ \partial' \in \mathcal{D}_{\beta-\rho},\ \partial'' \in \mathcal{D}_{\gamma-\rho}. \tag{3.35}$$

Let $\beta, \gamma \in \Gamma$ such that $\{\alpha, \beta, \gamma\}$ is linearly independent. First we take $\partial' \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta \backslash \mathcal{D}_\gamma$ and $\partial'' \in \mathcal{D}_\alpha \cap \mathcal{D}_\gamma \backslash \{0\}$. Then ∂' and ∂'' are linearly independent. The last 3 terms in (3.34) vanish. So we obtain that the coefficient of ∂'' is zero, which implies

$$a_{\beta+\gamma} = a_\beta + a_\gamma. \tag{3.36}$$

By the above equation, we have

$$a_\beta = a_{\beta-\gamma+\gamma} = a_{\beta-\gamma} + a_\gamma = a_\beta + a_{-\gamma} + a_\gamma, \tag{3.37}$$

which implies

$$a_{-\gamma} + a_\gamma = 0. \tag{3.38}$$

Thus (3.36) and (3.38) imply

$$\begin{aligned} a_{n\beta} &= a_{n\beta-\gamma+\gamma} \\ &= a_{n\beta-\gamma} + a_\gamma \\ &= a_\beta + a_{(n-1)\beta-\gamma} + a_\gamma \\ &= na_\beta + a_{-\gamma} + a_\gamma \\ &= na_\beta \end{aligned} \tag{3.39}$$

for $1 \le n \in \mathbb{Z}$. Taking $\gamma = \beta$, $\partial' \in \mathcal{D}_\alpha \cap \mathcal{D}_\beta \backslash \{0\}$ and $\partial'' \in \mathcal{D}_{\beta-\rho} \backslash \mathcal{D}_\alpha$ in (3.34), we obtain

$$a_\beta = \langle \partial_\beta, \beta \rangle. \tag{3.40}$$

Letting $\partial' \in \mathcal{D}_{\beta-\rho} \backslash \mathcal{D}_\alpha$ and $\partial'' \in \mathcal{D}_\alpha \cap \mathcal{D}_\gamma \backslash \mathcal{D}_\beta$ in (3.34), we get

$$(a_\gamma - \langle \partial_\beta, \gamma \rangle)\partial'' + \langle \partial'', \beta \rangle (\partial_\beta - \partial_{\beta+\gamma}) = 0 \tag{3.41}$$

by (3.36).

Taking the product (2.12) of (3.41) with α, β and γ, respectively, we obtain

$$\langle \partial_\beta, \alpha \rangle = \langle \partial_{\beta+\gamma}, \alpha \rangle, \quad a_\gamma - \langle \partial_\beta, \gamma \rangle = \langle \partial_{\beta+\gamma}, \beta \rangle - \langle \partial_\beta, \beta \rangle, \quad \langle \partial_\beta, \gamma \rangle = \langle \partial_{\beta+\gamma}, \gamma \rangle. \tag{3.42}$$

By (3.40) and the last two equations of (3.42), we get

$$a_\beta - \langle \partial_\gamma, \beta \rangle = \langle \partial_{\beta+\gamma}, \gamma \rangle - \langle \partial_\gamma, \gamma \rangle = \langle \partial_\beta, \gamma \rangle - a_\gamma. \tag{3.43}$$

Replacing γ by $\gamma - \beta$ in the first equation of (3.42), we have

$$\langle \partial_\beta, \alpha \rangle = \langle \partial_\gamma, \alpha \rangle \text{ for } \beta, \gamma \in \Gamma \text{ such that } \{\alpha, \beta, \gamma\} \text{ is linearly independent.} \tag{3.44}$$

Taking the product (2.12) of (3.34) with α and $\partial', \partial'' \notin \mathcal{D}_\alpha$, we obtain

$$a_\gamma - \langle \partial_\beta, \gamma \rangle = a_\beta - \langle \partial_\gamma, \beta \rangle \tag{3.45}$$

by (3.44). Comparing (3.45) with (3.43), we have that the first term of (3.41) is zero. Thus replacing γ by $\gamma - \beta$ in (3.41), we get

$$\partial_\beta = \partial_\gamma \text{ for all } \beta, \gamma \in \Gamma \text{ such that } \alpha, \beta, \gamma \text{ are linearly independent.} \tag{3.46}$$

For any given $\beta,\gamma\in\Gamma\backslash\mathbb{F}\alpha$, we can choose $\delta\in\Gamma$ such that $\{\alpha,\beta,\delta\}$ and $\{\alpha,\gamma,\delta\}$ are linearly independent. Thus we have

$$\partial_\beta=\partial_\delta=\partial_\gamma. \tag{3.47}$$

Hence ∂_β does not depend on $\beta\in\Gamma\backslash\mathbb{F}\alpha$, which is now denoted by $\tilde{\partial}$. Moreover, (3.33) and (3.40) imply

$$e_{\beta,\vec{j}}\partial'=\langle\tilde{\partial},\beta\rangle\partial'-\langle\partial',\alpha\rangle\tilde{\partial}\ \ \text{for}\ \ \beta\in\Gamma\backslash\mathbb{F}\alpha,\ \partial'\in\mathcal{D}_{\beta-\rho}. \tag{3.48}$$

For any $\gamma=b\alpha\in\Gamma$ and $\partial'\in\mathcal{D}_{\gamma-\rho}$, (3.10) and (3.48) show

$$\begin{aligned}&\langle\beta(\tilde{\partial})\partial'-\alpha(\partial')\tilde{\partial},\gamma\rangle\partial''-\langle\partial'',\alpha+\beta\rangle(\beta(\tilde{\partial})\partial'-\alpha(\partial')\tilde{\partial})\\&+\langle\partial',\alpha+\gamma\rangle e_{\gamma,\vec{k}}\partial''-\langle e_{\gamma,\vec{k}}\partial'',\beta\rangle\partial'\\=\ &\langle\tilde{\partial},\beta+\gamma\rangle(\gamma(\partial')\partial''-\beta(\partial'')\partial')-(\gamma(\partial')\alpha(\partial'')-\beta(\partial'')\alpha(\partial'))\tilde{\partial},\end{aligned} \tag{3.49}$$

equivalently,

$$-\langle e_{\gamma,\vec{k}}\partial''-\gamma(\tilde{\partial})\partial''+\alpha(\partial'')\tilde{\partial},\beta\rangle\partial'+(b+1)\alpha(\partial')(e_{\gamma,\vec{k}}\partial''-\gamma(\tilde{\partial})\partial''+\alpha(\partial'')\tilde{\partial})=0. \tag{3.50}$$

Since $\partial'\in\mathcal{D}_{\beta-\rho}$ is arbitrary, we obtain the coefficient of ∂':

$$\langle e_{\gamma,\vec{k}}\partial''-\gamma(\tilde{\partial})\partial''+\alpha(\partial'')\tilde{\partial},\beta\rangle=0. \tag{3.51}$$

By (2.12) and the arbitrariness of $\beta\in\Gamma\backslash\mathbb{F}\alpha$, (3.51) implies

$$e_{\gamma,\vec{k}}\partial''-\gamma(\tilde{\partial})\partial''+\alpha(\partial'')\tilde{\partial}=0\ \ \text{for}\,\partial''\in\mathcal{D}_{\gamma-\rho},\ \ \gamma\in\Gamma\cap\mathbb{F}\alpha. \tag{3.52}$$

This together with (3.48) imply (3.30).

Now we go back to (3.13). If $\alpha\neq\rho$, then $\tilde{\partial}\in\mathcal{D}_{\alpha-\rho}$ and there exist $u'=\overline{x}^{\alpha,\vec{i}}\tilde{\partial}\in\mathcal{S}_\alpha^{[\vec{i}]}$. It remains to consider the case $\alpha=\rho$.

Claim 3. If $\alpha=\rho$ and $i_{\ell_2}\neq0$, then $\chi_{\ell_2}(\tilde{\partial})=0$.

By assumption, $\rho=\alpha\neq0$ and $i_{\ell_2}\neq0$ imply that $\ell_1=0$ and $\ell_2>0$. By (3.30), $e_{\beta,\vec{j}}$ does not depend on $\vec{j}$, which is simply denoted by e_β. First we suppose that there exists $p\neq\ell_2$ such that $\rho_p\neq0$. Take

$$u_1=D_{p,\ell_2}(t_{\ell_2})=t_{\ell_2}\partial'+\partial_p,\ \text{where}\ \partial'=\rho_p\partial_{\ell_2}-\rho_{\ell_2}\partial_p. \tag{3.53}$$

We claim that the leading degree of $d(u_1)$ is $\leq\vec{i}$. Otherwise, let $x^{\rho,\vec{m}}\partial$ be the leading term of $d(u_1)$ with $\vec{m}>\vec{i}$ and $\partial\in\mathcal{D}_0\backslash\{0\}$. Take

$$u_2=x^\rho\partial'',\ \text{where}\ \partial''=\partial_p\ \text{if}\ \rho(\partial)=0\ \text{or}\ \partial''\in\mathcal{D}_0\cap\mathcal{D}_\rho\backslash\{0\}\ \text{otherwise}. \tag{3.54}$$

Then $[d(u_1), u_2]$ has a term $x^{2\rho,\vec{m}}(\rho(\partial)\partial'' - \rho(\partial'')\partial) \notin \mathcal{S}^{[\vec{i}]}$. On the other hand, $[u_1, u_2] \in \mathcal{S}^{[0]}$. So by (3.6), $d([u_1, u_2]) \in \mathcal{S}^{[\vec{i}]}$ and $d(u_2) \in \mathcal{S}^{[\vec{i}]}_{2\rho}$. One can verify that $[u_1, \mathcal{S}^{[\vec{i}]}_{2\rho}] \subset \mathcal{S}^{[\vec{i}]}_{2\rho}$. This leads a contradiction. Hence we can assume

$$d(u_1) = x^{\rho,\vec{i}}\hat{\partial} + \sum_{\vec{n}<\vec{i}} x^{\rho,\vec{n}}\partial^{(\vec{n})} \in \mathcal{S}_\rho, \tag{3.55}$$

with $\hat{\partial}, \partial^{(\vec{n})} \in \mathcal{D}$. For any $\beta \in \Gamma$ with $\beta(\partial') = 0$, we take $\partial'' \in \mathcal{D}_{\beta-\rho}$. By (3.8) and considering the term with degree $\vec{i}$ in $d([u_1, x^\beta\partial''])$, we obtain

$$e_\beta(\beta_p\partial'' - \chi_{\ell_2}(\partial'')\partial') = \langle\hat{\partial}, \beta\rangle\partial'' - \langle\partial'', \rho\rangle\hat{\partial} + (i_{\ell_2}\rho_p + \beta_p)e_\beta\partial'' - \chi_{\ell_2}(e_\beta\partial'')\partial'. \tag{3.56}$$

Applying (3.30) in (3.56) and noting that $\rho(\partial') = 0$, we have

$$\begin{aligned} & -\chi_{\ell_2}(\partial'')\beta(\tilde{\partial}')\partial' \\ = \;& \beta(\hat{\partial})\partial'' - \rho(\partial'')\hat{\partial} + (i_{\ell_2}+1)\rho_p(\beta(\tilde{\partial})\partial'' - \rho(\partial'')\tilde{\partial}) \\ & -(\beta(\tilde{\partial})\chi_{\ell_2}(\partial'') - \rho(\partial'')\chi_{\ell_2}(\tilde{\partial}))\partial', \end{aligned} \tag{3.57}$$

equivalently,

$$((i_{\ell_2}+1)\rho_p\beta(\tilde{\partial})+\beta(\hat{\partial}))\partial'' = \rho(\partial'')(\hat{\partial}+(i_{\ell_2}+1)\rho_p\tilde{\partial}-\chi_{\ell_2}(\tilde{\partial})\partial') \;\text{ for } \beta \in \Gamma,\ \partial'' \in \mathcal{D}_{\beta-\rho}. \tag{3.58}$$

Since $\partial'' \in \mathcal{D}_{\beta-\rho}$ is arbitrary, the coefficient of ∂'' must be zero, which implies that the right-hand side in (3.58) is zero. We take $\beta = \rho$ and $\partial'' = \partial_p$ such that $\rho(\partial'') \neq 0$. Applying χ_{ℓ_2} to the right hand side of (3.58) and noting that $\chi_{\ell_2}(\partial') = \rho_p$, we obtain

$$\chi_{\ell_2}(\tilde{\partial}) = -(i_{\ell_2}\rho_p)^{-1}\chi_{\ell_2}(\hat{\partial}) = 0, \tag{3.59}$$

where the last equality follows from (3.55) and Lemma 2.2 and the assumption that $i_{\ell_2} \neq 0$.

Next we assume that $\rho_p = 0$ for all $p \in \overline{1,\ell}\backslash\{\ell_2\}$. Then $\rho_{\ell_2} \neq 0$. If $\ell_2 \geq 2$, we take $u_1 = D_{\ell_2,1}(t_1) = \rho_{\ell_2}t_1\partial_1 + \partial_{\ell_2}$. For any $\beta \in \Gamma$ with $\beta_{\ell_2} = 0$ and any $\partial'' \in \mathcal{D}_{\beta-\rho}$, we have

$$e_\beta(\beta_{\ell_2}\partial'' - \chi_1(\partial'')\rho_{\ell_2}\partial_1) = \langle\hat{\partial}, \beta\rangle\partial'' - \langle\partial'', \rho\rangle\hat{\partial} + (\rho_{\ell_2} + \beta_{\ell_2})e_\beta\partial'' - \chi_1(e_\beta\partial'')\rho_{\ell_2}\partial_1 \tag{3.60}$$

by the above arguments and considering $d([u_1, x^\beta\partial''])$.

By (3.30) and a similar argument as that for (3.57), the coefficient of ∂'' must be zero. Then by calculating coefficient of ∂_{ℓ_2}, we obtain that $\chi_{\ell_2}(\tilde{\partial}) = -\rho_{\ell_2}^{-1}\chi_{\ell_2}(\hat{\partial}) = 0$.

Finally, we assume that $\ell_2 = 1$ ($\ell_1 = 0$) and $\rho_p = 0$ for all $p \in \overline{2,\ell}$. Then $\rho_1 \neq 0$, say $\rho_1 = 1$. For any $\partial \in \mathcal{D}$, we have $\chi_1(\partial) = \rho(\partial)$. In this case, $\mathbb{N}^{\ell_1+\ell_2} = \mathbb{N}$ and $\vec{j} = j$ for all $\vec{j} \in \mathbb{N}^{\ell_1+\ell_2}$. For any $\beta \in \Gamma$, we define the linear map $\overline{e}_\beta : \mathcal{D}_{\beta-\rho} \to \mathcal{D}$ by

$$d(x^\beta\partial') \equiv x^{\rho+\beta,i}e_\beta\partial' + x^{\rho+\beta,i-1}\overline{e}_\beta\partial' \pmod{\mathcal{S}^{(i-2)}_{\rho+\beta}} \quad \text{for } \partial' \in \mathcal{D}_{\beta-\rho} \tag{3.61}$$

(cf. (2.19)). By calculating the term of $d([x^\beta\partial', x^\gamma\partial''])$ with degree $i-1$, we obtain

$$\begin{aligned} & \langle\overline{e}_\beta\partial',\gamma\rangle\partial'' - (i\chi_1(\partial'')e_\beta\partial' + \langle\partial'',\rho+\beta\rangle\overline{e}_\beta\partial') \\ & +(i\chi_1(\partial')e_\gamma\partial'' + \langle\partial',\rho+\gamma\rangle\overline{e}_\gamma\partial'') - \langle\overline{e}_\gamma\partial'',\beta\rangle\partial' \\ = \; & \overline{e}_{\beta+\gamma}(\gamma(\partial')\partial'' - \beta(\partial'')\partial') \end{aligned} \tag{3.62}$$

for $\beta,\gamma\in\Gamma,\ \partial'\in\mathcal{D}_{\beta-\rho}$ and $\partial''\in\mathcal{D}_{\gamma-\rho}$ (cf. (3.10)). Write $\overline{e}_\beta\partial'$ as $\overline{e}_\beta\partial' = \hat{e}_\beta\partial' - i\beta(\hat{\partial})\partial'$ for all $\beta\in\Gamma$ and $\partial'\in\mathcal{D}_{\beta-\rho}$. By (3.30) and the fact $\chi_1(\partial)=\rho(\partial)$, we observe that $\hat{e}_\beta$ satisfies exactly the same equation as e_β in (3.10). The only difference between $\hat{e}_\beta$ and e_β is that $\hat{e}_\beta$ has codomain $\mathcal{D}$, while e_β has codomain $\mathcal{D}_\beta$ (cf. (3.7) with $\alpha=\rho$). However, this fact was not used when we used (3.10) in the proof of (3.30) if $\ell_1=0$. Therefore, we conclude that there exists $\hat{\partial}\in\mathcal{D}$ such that

$$\hat{e}_\beta\partial' = \beta(\hat{\partial})\partial' - \rho(\partial')\hat{\partial} \qquad \text{for } \partial'\in\mathcal{D}_{\beta-\rho},\ \beta\in\Gamma. \tag{3.63}$$

Thus we can write (3.61) as

$$d(x^\beta\partial') \equiv x^{\rho+\beta,i}(\beta(\tilde{\partial})\partial'-\rho(\partial')\tilde{\partial})+x^{\rho+\beta,i-1}(\beta(\hat{\partial})\partial'-\rho(\partial')\hat{\partial}-i\beta(\tilde{\partial})\partial')\ (\mathrm{mod}\,\mathcal{S}^{(i-2)}_{\rho+\beta}). \tag{3.64}$$

Take $\beta\in\Gamma$ with $\beta_1\neq 0$. For any element

$$u = x^{\rho+\beta,j}\partial_u^{(j)} + x^{\rho+\beta,j-1}\partial_u^{(j-1)} + \cdots + x^{\rho+\beta}\partial_u^{(0)} \in \mathcal{S}^{[j]}_{\rho+\beta}, \quad j\geq 0, \tag{3.65}$$

we have

$$j\rho(\partial_u^{(j)}) + \beta(\partial_u^{(j-1)}) = 0. \tag{3.66}$$

Note that any such element is a linear combination of $D_{p,q}(x^{\rho+\beta,k})$ for $k\leq j$ and $p,q\in\overline{1,\ell}$. Write $u=u_1+u_2$ such that u_1 is a linear combination of $D_{p,q}(x^{\rho+\beta,j})$ and the leading degree of u_2 is $\leq j-1$. Then by Lemma 2.2, $\beta(\partial_{u_2}^{(j-1)})=0$, and we can assume

$$u_1 = D_{p,q}(x^{\rho+\beta,j}) = x^{\rho+\beta,j}(\beta_q\partial_p - \beta_p\partial_q) + jx^{\rho+\beta,j-1}(\delta_{q,1}\partial_p - \delta_{p,1}\partial_q). \tag{3.67}$$

By the fact $\rho_p=\delta_{p,1}, p\in\overline{1,\ell}$, we obtain (3.66). Applying the result (3.66) to (3.64), we have

$$i(\beta(\tilde{\partial})\rho(\partial')-\rho(\partial')\rho(\tilde{\partial}))+(\beta(\hat{\partial})\beta(\partial')-\rho(\partial')\beta(\hat{\partial})-i\beta(\tilde{\partial})\beta(\partial'))=0 \ \text{ for } \partial'\in\mathcal{D}_{\beta-\rho}. \tag{3.68}$$

By the fact $\beta(\partial')=\rho(\partial')$ and taking $\partial'\in\mathcal{D}_{\beta-\rho}\backslash\mathcal{D}_\rho$, we obtain $\rho(\tilde{\partial})=0$, that is, $\chi_{\ell_2}(\tilde{\partial})=0$. This proves Claim 3.

Next by our last claim, we can take some $u=\overline{x}^{\rho,\vec{i}}\tilde{\partial}\in\mathcal{S}_\rho$ if $\vec{i}\neq\vec{0}$ and $u=x^\rho\tilde{\partial}\in\mathcal{W}^{[0]}_\rho$ if $\vec{i}=\vec{0}$. This completes the proof of Lemma 3.2. $\qquad\Box$

Now we shall describe homogeneous derivations of degree 0. Recall the derivations ∂_{t_i} and ∂_j^* on $\mathcal{A} = \mathcal{A}(\ell_1, \ell_2, \ell_3; \Gamma)$ defined in (1.11) for $i \in \overline{1, \ell_1 + \ell_2}$ and $j \in \overline{1, \ell_2 + \ell_3}$. Set

$$\mathcal{D}^- = \sum_{i=1}^{\ell_2} \mathbb{F}\partial_{t_{\ell_1+i}}, \quad \mathcal{D}^+ = \sum_{j=1}^{\ell_2} \mathbb{F}\partial_j^*, \quad \mathcal{S}_0^+ = \mathcal{S}_0 + \mathcal{D} + \mathcal{D}^+ + \sum_{p=1}^{\ell_1} \mathbb{F}t_p\partial_p \tag{3.69}$$

(cf. (1.11)). Note that $\mathcal{D}^- \subset \mathcal{S}_0^+$ by the second equation in (1.13). Denote by $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$ the set of additive group homomorphisms from Γ to $\mathbb{F}$.

Lemma 3.3. *Let $d \neq 0$ be a derivation satisfying (3.6) with $\alpha = 0$. Then there exist $u \in \mathcal{S}_0^+$ and $\mu \in Hom_{\mathbb{Z}}(\Gamma, \mathbb{F})$ such that*

$$d(u') = [u, u'] + \mu(\beta)u' \qquad \text{for } u' \in \mathcal{S}_\beta. \tag{3.70}$$

Proof. Obviously, (3.70) defines a homogeneous derivation of $\mathcal{S}$ of degree 0. Denote such an derivation by $d_{u,\mu}$. We shall use the notations as before. In particular, $\vec{i}$ was defined in (3.7). However, it is possible that $\vec{i} \notin \mathbb{N}^{\ell_1+\ell_2}$; for instance, $\vec{i} = (-1, 0, \cdots, 0)$ for $d = \mathrm{ad}_{\partial_1}$.

For convenience, we shall assume $\ell_1 + \ell_2 \geq 1$ because the case $\ell_1 + \ell_2 = 0$ had been proved in [DZ] (cf. Theorem 4.2 there). Now (3.10) becomes

$$\langle e_{\beta,\vec{j}}\partial', \gamma\rangle\partial'' - \beta(\partial'')e_{\beta,\vec{j}}\partial' + \gamma(\partial')e_{\gamma,\vec{k}} - \langle e_{\gamma,\vec{k}}\partial'', \beta\rangle\partial' = e_{\beta+\gamma,\vec{j}+\vec{k}}(\gamma(\partial'')\partial' - \beta(\partial'')\partial'), \tag{3.71}$$

where $\beta, \gamma \in \Gamma$, $\partial' \in \mathcal{D}_{\beta-\rho}$ and $\partial'' \in \mathcal{D}_{\gamma-\rho}$ with the assumption (3.2).

Taking $\gamma = 0$ and $\vec{k} = 0$ in (3.71), we obtain $\langle e_{0,0}\partial'', \beta\rangle = 0$ for all $\beta \in \Gamma$. Hence,

$$e_{0,0}\partial'' \in \mathcal{D}_1 \qquad \text{for } \partial'' \in \mathcal{D}_\rho. \tag{3.72}$$

If $\partial'' \in \mathcal{D}_\rho \cap \mathcal{D}_3$, then for all $u = \overline{x}^{\beta,\vec{j}}\partial' \in \mathcal{S}_\beta$, we have

$$\beta(\partial'')d(u) = d([\partial'', u]) = [d(\partial''), u] + [\partial'', d(u)] = [d(\partial''), u] + \beta(\partial'')d(u). \tag{3.73}$$

due to that $\mathrm{ad}_{\partial''}$ acts as the scalar $\beta(\partial'')$ on $\mathcal{S}_\beta$. This shows that $d(\partial'')$ is in the center of $\mathcal{S}$. So

$$d(\partial'') = 0 \qquad \text{for } \partial'' \in \mathcal{D}_\rho \cap \mathcal{D}_3. \tag{3.74}$$

For any $\partial \in \mathcal{D}$, we define

$$p_\partial = \begin{cases} \max\{p \in \overline{1, \ell_1 + \ell_2} \,|\, a_p \neq 0\} & \text{if } \partial = \sum_{p=1}^{\ell} a_p\partial_p \in \mathcal{D}\backslash\mathcal{D}_3, \\ \ell + 1 & \text{otherwise.} \end{cases} \tag{3.75}$$

We shall choose a basis B of $\mathcal{D}_\rho$ as follows. If $\rho \neq 0$, we take p to be the largest index with $\rho_{p-\ell_1} \neq 0$ when $\langle \mathcal{D}_3, \rho\rangle \neq 0$ and the smallest index with $\rho_{p-\ell_1} \neq 0$ when $\langle \mathcal{D}_3, \rho\rangle = 0$ (cf. (2.12)). Let

$$B = \begin{cases} \{\partial_q \in \mathcal{D}_0 \,|\, q \in \overline{1, \ell}\} & \text{if } \rho = 0, \\ \{\partial_q - \rho_{p-\ell_1}^{-1}\rho_{q-\ell_1}\partial_p \,|\, q \in \overline{1, \ell}\backslash\{p\}\} & \text{if } \rho \neq 0. \end{cases} \tag{3.76}$$

By this choice of basis, we observe that different elements $\partial \in B\backslash\mathcal{D}_3$ have different p_∂.

Claim. Replacing d by some $d - d_{u,\mu}$ if necessary, we can assume

$$e_{0,0}\partial'' = 0 \qquad \text{for } \partial'' \in \mathcal{D}_\rho. \tag{3.77}$$

Suppose $\partial'' \in B$ such that $e_{0,0}\partial'' \neq 0$ with $p_{\partial''}$ as minimal as possible. In this case, (3.72) and (3.74) force $\ell_1 \geq 1, \partial'' \notin \mathcal{D}_3$ and $p = p_{\partial''} \in \overline{1, \ell_1 + \ell_2}$. We shall prove the claim by induction on p. For $\partial' \in \mathcal{D}_{\beta-\rho}$ and any $\beta \in \Gamma$ such that $\beta(\partial'') = 0$, we have

$$[\partial'', \bar{x}^{\beta,\vec{j}}\partial'] = j_p \bar{x}^{\beta,\vec{j}-1_{[p]}}\partial'. \tag{3.78}$$

We want to prove that if $e_{0,0}\partial'' \neq 0$, then we must have

$$i_q = 0 \qquad \text{for } q \in \overline{p_{\partial''} + 1, \ell_1 + \ell_2} \ \text{ and } \ e_{0,0}\partial'' \in \mathcal{D}^{(p)} = \mathrm{Span}\{\partial_1, ..., \partial_p\} \tag{3.79}$$

(cf. (3.7) for $\vec{i}$). If $i_q \neq 0$ for some $q > p$, then $\ell_1 + \ell_2 \geq 2$, and so $\partial_q \in \mathcal{D}_{\rho-\rho}$ (cf. (2.13)). Taking $\beta = \rho$ and $\partial' = \partial_q$ in (3.78), we have $[\partial'', x^\rho\partial_q] = 0$ and $[\partial'', d(x^\rho\partial_q)] \in \mathcal{S}_\rho^{[\vec{i}-1_{[p]}]}$. However, the term with degree $\vec{i}-1_{[q]}$ in $[d(\partial''), x^\rho\partial_q]$ is $-i_q x^{\rho,\vec{i}-1_{[q]}}e_{0,0}\partial'' \notin \mathcal{S}^{[\vec{i}-1_{[p]}]}$, which leads a contradiction. Similarly, if $\chi_q(e_{0,0}\partial'') \neq 0$ for some $q \in \overline{p+1, \ell_1}$, then $[d(\partial''), x^{\rho,1_{[q]}}\partial_1]$ has a term $\chi_q(e_{0,0}\partial'')x^{\rho,\vec{i}}\partial_1 \in \mathcal{S}_\rho^{[\vec{i}]}\backslash\mathcal{S}_\rho^{(\vec{i})}$ and $[\partial'', x^{\rho,1_{[q]}}\partial_1] = 0$. But the leading degree of $[\partial'', d(x^{\rho,1_{[q]}}\partial_1)]$ is $\leq \vec{i} - 1_{[p]} + 1_{[q]} < \vec{i}$, which leads a contradiction again. This proves (3.79).

Since $e_{0,0}\partial'' \neq 0$, we must have $\vec{i} \in \mathbb{N}^{\ell_1+\ell_2}$ because otherwise $\mathcal{S}^{[\vec{i}]} = \mathcal{S}^{(\vec{i})}$, and we have $e_{\beta,0}\partial'' = 0$ for all $\beta \in \Gamma$ by (3.8). Thus $i_q \geq 0$ for all $q \in \overline{1, \ell}$.

Let $p' = p_{e_{0,0}\partial''}$ (cf. (3.75)). By (3.72) and (3.79), $p' \leq \min\{p, \ell_1\}$. Write $e_{0,0}\partial'' = \sum_{q=1}^{p'} a_q\partial_q$. According to the assumption (2.34), $\rho = 0$. If $p' < \ell_1 + \ell_2$, we let $k = i_{\ell_1+\ell_2} + 1 + \delta_{p,\ell_1+\ell_2}$ and take

$$\begin{aligned} u_1 &= k^{-1}\sum_{q=1}^{p'} a_q D_{q,\ell_1+\ell_2}(t^{\vec{i}+1_{[p]}+1_{[\ell_1+\ell_2]}}) \\ &= k^{-1}\sum_{q=1}^{p'} a_q(kt^{\vec{i}+1_{[p]}} - (i_q + \delta_{q,p})t^{\vec{i}+1_{[p]}+1_{[\ell_1+\ell_2]}-1_{[q]}}) \\ &= t^{\vec{i}+1_{[p]}}e_{0,0}\partial'' - k^{-1}\sum_{q=1}^{p'} a_q(i_q + \delta_{q,p})t^{\vec{i}+1_{[p]}+1_{[\ell_1+\ell_2]}-1_{[q]}}. \end{aligned} \tag{3.80}$$

If $p' = \ell_1 + \ell_2$, we must have $p' = p = \ell_1$ and $\ell_2 = 0$ due to $p' \leq \ell_1$. Moreover, (2.11) and Lemma 2.2 shows $\ell_1 \geq 2$ and $i_{\ell_1} = 0$. By our choice of B in (3.76), we obtain $\partial'' = \partial_{\ell_1}$. According to the assumption of p, we get $e_{0,0}\partial_q = 0$ for $q \in \overline{1, \ell_1 - 1}$. We want to prove $\vec{i} = 0$. Suppose that $q' < \ell_1$ is the maximal index such that $i_{q'} \neq 0$. Applying d to

$$(\delta_{q,q'} - \delta_{q,\ell_1})\partial_q = [\partial_q, t_{q'}\partial_{q'} - t_{\ell_1}\partial_{\ell_1}] \quad \text{for } q \in \overline{q', \ell_1} \tag{3.81}$$

and calculating the term with degree $\vec{i}$, we obtain

$$(\delta_{q,q'}-\delta_{q,\ell_1})e_{0,0}\partial_q=\chi_{q'}(e_{0,0}\partial_q)\partial_{q'}-\chi_{\ell_1}(e_{0,0}\partial_q)\partial_{\ell_1}-i_{q'}e_{0,0}\partial_q+(i_q+1)\partial^{(\vec{i}+1_{[q]})}_{d(u_2)}, \quad (3.82)$$

where $\partial^{(\vec{i}+1_{[q]})}_{d(u_2)}$ is the element in $\mathcal{D}$ such that the term of $d(u_2)$ with degree $\vec{i}+1_{[q]}$ is $t^{\vec{i}+1_{[q]}}\partial^{(\vec{i}+1_{[q]})}_{d(u_2)}$, and $u_2=t_{q'}\partial_{q'}-t_{\ell_1}\partial_{\ell_1}$. Since $e_{0,0}\partial_q=0$ for $q<\ell_1$, by (3.82), we have

$$\partial^{(\vec{i}+1_{[q]})}_{d(u_2)}=0 \qquad \text{for } q\in\overline{q',\ell_1-1}. \quad (3.83)$$

Since $i_q=0$ for $q\in\overline{q'+1,\ell_1}$, we get

$$\{\vec{j}\in\mathbb{N}^{\ell_1+\ell_2}\mid \vec{i}+1_{[q']}>\vec{j}\geq\vec{i}+1_{[\ell_1]}\}=\{\vec{i}+1_{[q]}\mid q\in\overline{q'+1,\ell_1}\}. \quad (3.84)$$

Thus by (3.83) and (3.84), we can assume

$$d(t_{q'}\partial_{q'}-t_{\ell_1}\partial_{\ell_1})\equiv t^{\vec{i}+1_{[\ell_1]}}\hat{\partial}\pmod{\mathcal{S}^{[\vec{i}]}} \quad (3.85)$$

(cf. (2.20)), where $\hat{\partial}=\partial^{(\vec{i}+1_{[\ell_1]})}_{d(u_2)}\in\mathcal{D}_0$. Now letting $q=\ell_1$ in (3.82) and applying χ_{ℓ_1} to it (cf. (2.26)), we have

$$-\chi_{\ell_1}(e_{0,0}\partial_{\ell_1})=-\chi_{\ell_1}(e_{0,0}\partial_{\ell_1})-i_{q'}\chi_{\ell_1}(e_{0,0}\partial_{\ell_1})+\chi_{\ell_1}(\hat{\partial}). \quad (3.86)$$

Thus by (3.85), (3.86) and Lemma 2.2, we obtain $a_{\ell_1}=i_{q'}^{-1}\chi_{\ell_1}(\hat{\partial})=0$, which contradicts the case $p'=\ell_1$. Therefore $\vec{i}=0$. In this case, we take

$$u_1=t_{\ell_1}e_{0,0}\partial''=a_{\ell_1}t_{\ell_1}\partial_{\ell_1}+\sum_{q=1}^{\ell_1-1}a_qt_{\ell_1}\partial_q\in\mathcal{S}_0^+. \quad (3.87)$$

Now it is straightforward to verify that with the choices of u_1 in (3.80) and (3.84), one has

$$u_1\in\mathcal{S}_0^+,\ [u_1,\mathcal{S}^{[\vec{j}]}]\subset\mathcal{S}^{[\vec{i}+\vec{j}]} \text{ for } \vec{j}\in\mathbb{N}^{\ell_1+\ell_2} \text{ and } [u_1,\partial]\subset\mathcal{S}^{(\vec{i})} \quad (3.88)$$

for $\partial\in B$ with $p_\partial<p$. Replacing d by $d+(i_p+1)^{-1}\mathrm{ad}_{u_1}$, we have $e_{0,0}\partial=0$ for all $\partial\in B$ with $p_\partial\leq p$, which implies our claim (3.77) by induction on $p_{\partial''}$.

Suppose $i_p<0$ for some $p\in\overline{1,\ell_1+\ell_2}$. Then for any $\vec{j}\in\mathbb{N}^{\ell_1+\ell_2}$ with $j_p<-i_p$, we have $\mathcal{S}^{[\vec{i}+\vec{j}]}=\mathcal{S}^{(\vec{i}+\vec{j})}$. Thus

$$e_{\beta,\vec{j}}=0 \qquad \text{for } \beta\in\Gamma,\ \vec{j}\in\mathbb{N}^{\ell_1+\ell_2} \text{ with } j_p<-i_p \text{ if } i_p<0. \quad (3.89)$$

Since $\ell\geq 3$, any element $\bar{x}^{\beta,\vec{k}}\partial'\in\mathcal{S}$ can be generated by $\{\bar{x}^{\beta,\vec{j}}\partial'\mid\beta\in\Gamma,\ \vec{j}\in\mathbb{N}^{\ell_1+\ell_2}$ with $j_p<2\}$ for each $p\in\overline{1,\ell_1+\ell_2}$ by induction on k_p. This together with (3.89) shows that if $i_p<0$, then $i_p=-1$ and $i_q\geq 0$ for any $q\in\overline{1,\ell_1+\ell_2}\backslash\{p\}$.

For any $\partial'' \in \mathcal{D}_\rho$, we denote $p = p_{\partial''}$. Let $\beta \in \Gamma$ with $\beta(\partial'') = 0$ and let $\partial' \in \mathcal{D}_{\beta-\rho} \cap \mathcal{D}_\rho$. Applying d to (3.78), calculating the term with degree $\vec{i} + \vec{j} - 1_{[p]}$ and using (3.77), we obtain

$$\begin{aligned} j_p e_{\beta,\vec{j}-1_{[p]}} \partial' &= \langle \partial_{d(\partial'')}^{(\vec{i}-1_{[p]})}, \beta \rangle \partial' + \sum_{q=p+1}^{\ell_1+\ell_2} j_q \chi_q(\partial_{d(\partial'')}^{(\vec{i}-1_{[p]}+1_{[q]})}) \partial' \\ &\quad - \sum_{q=p+1}^{\ell_1+\ell_2} \chi_q(\partial')(i_q+1) \partial_{d(\partial'')}^{(\vec{i}-1_{[p]}+1_{[q]})} + (i_p + j_p) e_{\beta,\vec{j}} \partial', \end{aligned} \tag{3.90}$$

where $\partial_{d(\partial'')}^{(\vec{i}-1_{[p]})}, \partial_{d(\partial'')}^{(\vec{i}-1_{[p]}+1_{[q]})}$ are the elements in $\mathcal{D}$ such that the terms of $d(\partial'')$ with degrees $\vec{i} - 1_{[p]}$, $\vec{i} - 1_{[p]} + 1_{[q]}$ are $t^{\vec{i}-1_{[p]}} \partial_{d(\partial'')}^{(\vec{i}-1_{[p]})}$, $t^{\vec{i}-1_{[p]}+1_{[q]}} \partial_{d(\partial'')}^{(\vec{i}-1_{[p]}+1_{[q]})}$, respectively. Note that if $p = p_{\partial''} = \ell + 1$, this equation is trivial since all terms are zero. Observe that on the right-hand side of (3.90), the second term does not depend on β and the third terms does not depend on $\beta, \vec{j}$. Moreover, $\partial_{d(\partial'')}^{(\vec{i}-1_{[p]})} = 0$ if $i_p \leq 0$ or $i_{q'} < 0$ for some $q' \neq p$ and $\partial_{d(\partial'')}^{(\vec{i}-1_{[p]}+1_{[q]})} = 0$ if $i_p \leq 0$ or $i_{q'} < 0$ for some $q' \neq q$.

Using the arguments as those in the above and replacing d by some $d - d_{u,\mu}$, we can assume

$$e_{0,\vec{k}} \partial'' = 0 \qquad \text{for } \partial'' \in \mathcal{D}_\rho, \ \vec{k} \in \mathbb{N}^{\ell_1+\ell_2}. \tag{3.91}$$

Then setting $\beta = 0$ in (3.90) and using (3.91), we see that the sum of the second term and the third term of the right-hand side is zero. Thus

$$j_p e_{\beta,\vec{j}-1_{[p]}} \partial' = \langle \partial_{d(\partial'')}^{(\vec{i}-1_{[p]})}, \beta \rangle \partial' + (i_p + j_p) e_{\beta,\vec{j}} \partial' \tag{3.92}$$

for any $\beta \in \Gamma$, $\partial' \in \mathcal{D}_\rho \cap \mathcal{D}_{\beta-\rho}$, $\partial'' \in \mathcal{D}_\rho \cap \mathcal{D}_\beta$ with $p_{\partial''} = p$. By this, (3.71) and (3.91), we can show that if $\vec{i} \neq \vec{0}$, then there exists $u = \vec{t}^{\vec{i}} \tilde{\partial} \in \mathcal{S}_0$ such that

$$e_{\beta,\vec{j}} \partial' = \beta(\tilde{\partial}) \partial' \qquad \text{for } \beta \in \Gamma, \ \partial' \in \mathcal{D}_{\beta-\rho}. \tag{3.93}$$

Therefore the proof is completed by replacing d by some $d - \mathrm{ad}_u$ and the induction on $\vec{i}$.

It remains to consider $\vec{i} = 0$. In this case, by the proof of Theorem 4.2 in [DZ], there exist an additive group homomorphism $\mu : \Gamma \to \mathbb{F}$ and $\partial \in \mathcal{D}_2 + \mathcal{D}_3$ such that $d = d_{\partial,\mu}$. $\square$

Lemma 3.4. *Every homogeneous derivation $d \in (\mathrm{Der}\,\mathcal{S})_\alpha$ must satisfy the condition in (3.6).*

Proof. Choose $\Gamma' \subset \Gamma$ to be a nondegenerate subgroup of Γ generated by a finite subset Γ'_0 of Γ such that $\rho, \alpha \in \Gamma'_0$. Let $\mathcal{S}'$ be the Lie subalgebra of $\mathcal{S}$ generated by

$$\{D_{p,q}(x^{\beta,\vec{j}}) \mid \beta \in \Gamma'_0, \vec{j} \in \mathbb{N}^{\ell_1+\ell_2}, \ |\vec{j}| \leq 4, \ p, q \in \overline{1, \ell}\}. \tag{3.94}$$

Then it is straightforward to check that $\mathcal{S}' = \mathcal{S}(\ell_1, \ell_2, \ell_3; \rho, \Gamma')$, and $d' = d|_{\mathcal{S}'}$ is a homogeneous derivation of $\mathcal{S}'$ of degree α. Since (3.94) is a finite set and a derivation is determined by its action on generators, we see that the derivation d' of $\mathcal{S}'$ satisfies the condition in (3.6).

For $u \in \mathcal{S}$ and $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$, we use the notation $d_{u,\mu}$ to denote the derivation defined by the right-hand side of the equation (3.70). By Lemmas 3.1-3, there exist $u' = u'_1 + u'_2 \in \mathcal{S}'_\alpha + \mathcal{W}_\rho^{[0]}$ and $\mu' \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma', \mathbb{F})$ such that $d' = d_{u',\mu'}$, $u'_2 = 0$ if $\alpha \neq \rho$ and $\mu' = 0$ if $\alpha \neq 0$.

We claim that for $u \in \mathcal{S}$ and $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$, $d_{u,\mu}|_{\mathcal{S}_{\Gamma_1}} = 0$ if and only if $u \in \mathcal{D}_3$ and $\mu(\alpha) = -\alpha(u)$ for any $\alpha \in \Gamma_1$, where Γ_1 is any nondegenerate subgroup of Γ and $\mathcal{S}_{\Gamma_1} = \mathcal{S}(\ell_1, \ell_2, \ell_3; \rho, \Gamma_1)$. Write

$$u = \sum_{(\alpha,\vec{i}) \in K} x^{\alpha,\vec{i}} \partial^{(\alpha,\vec{i})}, \qquad \text{where } \partial^{(\alpha,\vec{i})} \in \mathcal{D} \text{ and} \tag{3.95}$$

$$K = \{(\alpha, \vec{i}) \in \Gamma \times \mathbb{N}^{\ell_1+\ell_2} \mid \partial^{(\alpha,\vec{i})} \neq 0\} \tag{3.96}$$

is a finite set. If some $(\alpha, \vec{i}) \in K$ with $\vec{i} \neq \vec{0}$ or $\alpha \notin \mathbb{F}\rho$, then clearly, there exists some $\partial \in \mathcal{D}_\rho$ such that

$$d_{u,\mu} = [u, \partial] = -\sum_{(\alpha,\vec{i}) \in K} \partial(x^{\alpha,\vec{i}}) \partial^{(\alpha,\vec{i})} \neq 0. \tag{3.97}$$

Thus we can rewrite

$$u = \sum_{c \in K'} x^{c\rho} \partial^{(c\rho)}, \qquad \text{where } K' = \{c \in \mathbb{F} \mid c\rho \in \Gamma, \partial^{(c\rho)} \neq 0\} \text{ is a finite set.} \tag{3.98}$$

If $\chi_p(\partial^{(c\rho)}) \neq 0$ for some $p \in \overline{1, \ell_1 + \ell_2}$, then we can choose $\partial \in \mathcal{D}_0 \backslash \{0\}$ with $\chi_p(\partial) = 0$. So $t_p \partial \in \mathcal{S}$ and

$$d_{u,\mu}(t_p \partial) = [u, t_p \partial] \neq 0. \tag{3.99}$$

Thus all $\partial^{(c\rho)} \in \mathcal{D}_3$ (cf. (2.1)). Let $0 \neq c \in K'$. Take $\beta \in \Gamma_1$ with $\beta(\partial^{(c\rho)}) \neq 0$ and choose $\partial \in \mathcal{D}_\beta \cap \mathcal{D}_\rho$. Then we have

$$d_{u,\mu}(x^\beta \partial) = \sum_{c \in K'} \beta(\partial^{(c\rho)}) x^{c\rho+\beta} \partial^{(c\rho)} + \mu(\beta) x^\beta \partial \neq 0. \tag{3.100}$$

Hence $u = \partial^{(0)} \in \mathcal{D}_3$. Then for any $x^\alpha \partial \in \mathcal{S}_{\Gamma_1}$, we have

$$0 = d_{u,\mu}(x^\alpha \partial) = (\alpha(u) + \mu(\alpha)) x^\alpha \partial, \tag{3.101}$$

which implies $\mu(\alpha) = -\alpha(u)$ for all $\alpha \in \Gamma_1$. This proves the claim.

We can regard $\mathcal{D}_3$ as a subspace of $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$. Choose a subspace $\mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$ of $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$ such that $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F}) = \mathcal{D}_3 \oplus \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$ as vector spaces. Thus we can

always assume $\mu \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$ when we use the notation $d_{u,\mu}$. Hence for any $u \in \mathcal{S}_\alpha + \mathcal{W}_\rho^{[0]}$ and $\mu \in \mathrm{Hom}^*_{\mathbb{Z}}(\Gamma, \mathbb{F})$, $d_{u,\mu}|_{\mathcal{S}_{\Gamma_1}} = 0$ implies $u = 0$ and $\mu|_{\Gamma_1} = 0$.

Let Γ_2 be the maximal subgroup of Γ such that for $\mathcal{S}_{\Gamma_2} = \mathcal{S}(\ell_1, \ell_2, \ell_3; \Gamma_2)$, there exists $\mu_2 \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma_2, \mathbb{F})$ with $d|_{\mathcal{S}_{\Gamma_2}} = d_{u_1,\mu_2}$ and $\mu_2|_{\Gamma_1} = \mu_1$.

Suppose $\Gamma_2 \neq \Gamma$. Take $\beta \in \Gamma \backslash \Gamma_2$. Let Γ_3 be the subgroup of Γ generated by Γ' and β. Then Γ_3 is finitely generated. Thus there exist u_3, μ_3 such that $d|_{\mathcal{S}_{\Gamma_3}} = d_{u_3,\mu_3}$. Then we have $d_{u_3-u_1,\mu_3-\mu_1}|_{\mathcal{S}_{\Gamma'}} = d|_{\mathcal{S}_{\Gamma'}} - d|_{\mathcal{S}_{\Gamma'}} = 0$. Thus $u_3 = u_1$ and $\mu_3|_{\Gamma'} = \mu_1|_{\Gamma'}$. Similarly, $\mu_2|_{\Gamma_2\cap\Gamma_3} = \mu_3|_{\Gamma_2\cap\Gamma_3}$.

Let Γ_4 be the subgroup of Γ generated by Γ_2 and β. Define $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma_4, \mathbb{F})$ as follows. For any $\gamma \in \Gamma_4$, we can write $\gamma = \tau + n\beta$ with $n \in \mathbb{Z}$ and $\tau \in \Gamma_2$. Define $\mu_4(\gamma) = \mu_2(\tau) + n\mu_3(\beta)$. Suppose $\tau + n\beta = 0$ for some $n \in \mathbb{Z}$. Then $\tau = -n\beta \in \Gamma_2 \cap \Gamma_3$. Since $\mu_2|_{\Gamma_2\cap\Gamma_3} = \mu_3|_{\Gamma_2\cap\Gamma_3}$, we have $\mu_2(\tau) = \mu_3(\tau) = \mu_3(-n\beta)$. But obviously, $\mu_3(-n\beta) = -n\mu_3(\beta)$. Hence $\mu_2(\tau) + n\mu_3(\beta) = 0$. This shows that $\mu_4 \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma_4, \mathbb{F})$ is uniquely defined. So $d|_{\mathcal{S}_{\Gamma_4}} = d_{u_1,\mu_4}$ and $\Gamma_4 \supset \Gamma_2$, $\Gamma_4 \neq \Gamma_2$. This contradicts the maximality of Γ_2. Therefore, $\Gamma_2 = \Gamma$ and $d = d_{u_1,\mu_2}$ satisfies the condition in (3.6). $\square$

Lemma 3.5. *Let d be any derivation of $\mathcal{S}$. Write*

$$d = \sum_{\alpha\in\Gamma} d_\alpha \qquad \text{with } d_\alpha \in (Der\,\mathcal{S})_\alpha. \tag{3.102}$$

Then

$$d_\alpha = 0 \ \text{ for all but a finite } \alpha \in \Gamma. \tag{3.103}$$

Proof. By Lemmas 3.1-4, for any $\alpha \in \Gamma\backslash\{0, \rho\}$, there exists $u_\alpha = \bar{x}^{\alpha,\vec{i}_\alpha}\partial_\alpha \in \mathcal{S}$ such that $d_\alpha = \mathrm{ad}_{u_\alpha}$. We always assume that $\partial_\alpha \neq 0$ if $d_\alpha \neq 0$ and $\vec{i}_\alpha$ is the leading degree of u_α. We shall prove that

$$Y = \{\alpha \in \Gamma\backslash\{0, \rho\} \mid \partial_\alpha \neq 0\} \tag{3.104}$$

is finite. Take a $\mathbb{F}$-basis $\{\alpha^{(\ell_1+1)}, ..., \alpha^{(\ell)}\}$ of $\mathbb{F}^{\ell_2+\ell_3}$ from $\Gamma\backslash\{\rho\}$. Define

$$Z' = \{\alpha \in Y \mid \partial_\alpha \in \mathcal{D}_1\}, \ \ Z_\beta = \{\alpha \in Y \mid \beta(\partial_\alpha) \neq 0\} \ \text{ for } \beta \in \Gamma. \tag{3.105}$$

Then

$$Y = Z' \bigcup \bigcup_{p\in\overline{\ell_1+1,\ell}} Z_{\alpha^{(p)}}. \tag{3.106}$$

So it is sufficient to prove that Z' is a finite set and so is Z_β for any $\beta \in \Gamma\backslash\{0, \rho\}$.

Suppose that Z' is an infinite set. Then $\ell_1 > 0$, and so we can assume $\rho = 0$ by Lemma 2.3. Hence $\mathcal{D}_0 \supset \mathcal{D}_2 + \mathcal{D}_3$, where $0 \in \Gamma$. Take $\partial \in \mathcal{D}_2 + \mathcal{D}_3$ such that there are infinite many $\alpha \in Z'$ with $\alpha(\partial) \neq 0$. For any $\alpha \in Z'$, we have

$$d_\alpha(\partial) \equiv -\alpha(\partial)\bar{x}^{\alpha,\vec{i}_\alpha}\partial \ (\mathrm{mod}\ \mathcal{S}_\alpha^{(\vec{i}_\alpha)}). \tag{3.107}$$

Thus there are infinitely many α with $d_\alpha(\partial) \neq 0$, which contradicts the fact that $d(\partial)$ is contained in a sum of finite number of $\mathcal{S}_\alpha$.

Assume that $\alpha \in Z_\beta$. For $\partial \in \mathcal{D}_{\beta-\rho}$, we have

$$d_\alpha(\overline{x}^{\beta,\vec{j}}\partial) \equiv \overline{x}^{\alpha+\beta,\vec{i}_\alpha+\vec{j}}(\beta(\partial_\alpha)\partial - \alpha(\partial)\partial_\alpha) \pmod{\mathcal{S}_{\alpha+\beta}^{(\vec{i}_\alpha+\vec{j})}} \tag{3.108}$$

(cf. (2.19)). The fact that $d(\partial)$ is contained in a sum of finite number of $\mathcal{S}_\alpha$ implies that

$$\beta(\partial_\alpha)\partial - \alpha(\partial)\partial_\alpha = 0 \tag{3.109}$$

for all but a finite $\alpha \in Z_\beta$. Since $\beta(\partial_\alpha) \neq 0$ by (3.105), $\partial \in \mathcal{D}_{\beta-\rho}$ is arbitrary and $\dim \mathcal{D}_{\beta-\rho} \geq 2$, (3.109) implies that $\partial_\alpha = 0$ for all but a finite $\alpha \in Z_\beta$. Therefore, Z_β is a finite set. $\square$

By Lemma 3.5, we have

$$\mathrm{Der}\,\mathcal{S} = \bigoplus_{\alpha\in\Gamma}(\mathrm{Der}\,\mathcal{S})_\alpha \tag{3.110}$$

(cf. (3.5)). For convenience, we identify an additive function μ with the derivation $d_{0,\mu}$ defined by μ. Thus

$$\mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{F}) \subset (\mathrm{Der}\,\mathcal{S})_0. \tag{3.111}$$

Recall the notations in (3.69). For any $\partial \in \mathcal{D}_2 + \mathcal{D}_3$, there is a unique way to decompose

$$\partial = \partial^+ + \partial^- \qquad \text{with } \ \partial^+ \in \mathcal{D}^+ + \mathcal{D}_3, \ \partial^- \in \mathcal{D}^-. \tag{3.112}$$

For any $\alpha \in \Gamma$, we define

$$\langle \partial^+, \alpha\rangle = \langle \partial, \alpha\rangle \tag{3.113}$$

(cf. (2.12)). Then $\mathcal{D}^+ + \mathcal{D}_3$ can be identified with a subspace of $\mathbb{F}$-linear function $\mathrm{Hom}_{\mathbb{F}}(\Gamma,\mathbb{F})$. Thus

$$\mathcal{D}^+ + \mathcal{D}_3 \subset \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{F}) \subset (\mathrm{Der}\,\mathcal{S})_0. \tag{3.114}$$

We shall identify u with $\mathrm{ad}_u|_{\mathcal{S}}$ for $u \in \mathcal{W}$ when the context is clear. In particular, for $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{F})$ and $u \in \mathcal{S}_\alpha$, we have

$$[\mu, u] = \mu(\alpha)u. \tag{3.115}$$

We summarize the results in Lemmas 3.1-3.5 as the following theorem.

Theorem 3.6. *The derivation algebra* $\mathrm{Der}\,\mathcal{S}$ *is an* Γ*-graded Lie algebra (cf. (3.5) and (3.110)) with*

$$(\mathrm{Der}\,\mathcal{S})_\alpha = \begin{cases} \mathcal{S}_\alpha & \text{if } \alpha \neq \rho, 0, \\ \mathcal{S}_\rho + \mathcal{W}_\rho^{[0]} & \text{if } \alpha = \rho \neq 0, \\ \mathcal{S}_0 + \mathbb{F}t_{\ell_1}\partial_{\ell_1} + \mathcal{D} + \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{F}) & \text{if } \alpha = 0. \end{cases} \tag{3.116}$$

4 Proof of the Main Theorem

In this section, we shall present the proof of the main theorem in this paper. First we need three more lemmas on derivations.

A linear transformation T on a vector space V is called *locally-nilpotent* if for any $v \in V$, there exists a positive integer n (depending on v) such that

$$T^n(v) = 0. \tag{4.1}$$

Lemma 4.1. *If $d \in \mathrm{Der}\,\mathcal{S}$ is locally-finite (cf. (1.16)), then*

$$d \in \mathcal{A}\mathcal{D}_1 + \mathcal{D} + \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F}). \tag{4.2}$$

If d is locally-nilpotent, then

$$d \in \mathcal{A}\mathcal{D}_1 \oplus \mathcal{D}^-. \tag{4.3}$$

Proof. If $\ell_2 + \ell_3 = 0$, there is nothing to prove. Suppose $\ell_2 + \ell_3 \geq 1$. Choose a total order $\leq$ on Γ compatible with its group structure. Let d be a locally-finite derivation. By Lemma 3.5, there exists a finite subset Γ_0 of Γ such that $d = \sum_{\alpha \in \Gamma_0} d_\alpha$. Write d as

$$d = u + \partial^- + \mu,\ u = \sum_{(\alpha,\vec{i}) \in \Gamma_0 \times \vec{J}_0} x^{\alpha,\vec{i}} \partial^{(\alpha,\vec{i})}, \tag{4.4}$$

where $\Gamma_0 \times \vec{J}_0$ is a finite subset of $\Gamma \times \mathbb{N}^{\ell_1+\ell_2}$ and $\partial^{(\alpha,\vec{i})} \in \mathcal{D}$, $\partial^- \in \mathcal{D}^-$, $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$. We want to prove

$$\begin{cases} \partial^{(\alpha,\vec{i})} \in \mathcal{D}_1 \ \text{ for } (0,0) \neq (\alpha,\vec{i}) \in \Gamma_0 \times \vec{J}_0 & \text{if } d \text{ is locally-finite;} \\ \partial^{(\alpha,\vec{i})} \in \mathcal{D}_1 \ \text{ for } (\alpha,\vec{i}) \in \Gamma_0 \times \vec{J}_0 \text{ and } \mu = 0 & \text{if } d \text{ is locally-nilpotent.} \end{cases} \tag{4.5}$$

We shall only give the proof of the first statement in (4.5). The proof of the second statement in (4.5) is similar. Suppose that $\partial^{(\delta,\vec{i})} \notin \mathcal{D}_1$ for some $\delta \in \Gamma_0 \backslash \{0\}$ (the case $\vec{i} \neq \vec{0}$ can be similarly proved). Take

$$\delta = \max\{\alpha \in \Gamma_0 \,|\, \partial^{(\alpha,\vec{i})} \notin \mathcal{D}_1 \text{ for some } \vec{i} \text{ with } (\alpha,\vec{i}) \in \Gamma_0 \times \vec{J}_0\} \tag{4.6}$$

and

$$\vec{j} = \max\{\vec{i} \in \vec{J}_0 \,|\, (\delta,\vec{i}) \in \Gamma_0 \times \vec{J}_0 \text{ with } \partial^{(\delta,\vec{i})} \notin \mathcal{D}_1\}. \tag{4.7}$$

Note that for any $\alpha \in \Gamma$ and $p \in \overline{1,\ell_1}$, $x^\alpha \partial_p$ is a locally-nilpotent derivation on $\mathcal{S}$. Hence $\exp(x^\alpha \partial_p)$ is an automorphism of $\mathcal{S}$. Let G be the subgroup of $\mathrm{Aut}(\mathcal{S})$ generated by such automorphisms. Elements in G induce automorphisms of $\mathrm{Der}\,\mathcal{S}$.

Claim 1. Replacing d by $g(d)$ (which is again locally-finite) for some $g \in G$, we can assume

$$j_p = 0 \quad \text{for } p \in \overline{1,\ell_1}, \tag{4.8}$$

where $\vec{j}$ is defined in (4.7).

We define another total order $>'$ on $\mathbb{N}^{\ell_1+\ell_2}$ different from (2.16) by

$$\vec{i} >' \vec{j} \iff \text{for the first } p \text{ with } i_p \neq j_p, \text{ we have } i_p > j_p. \tag{4.9}$$

With respect to this order, we define

$$\vec{k} = \max\{\vec{i} \in \vec{J_0} \,|\, \partial^{(\alpha,\vec{i})} \notin \mathcal{D}_1 \text{ for some } \alpha \in \Gamma_0 \text{ with } (\alpha,\vec{i}) \in \Gamma_0 \times \vec{J_0}\}, \tag{4.10}$$

$$\tau = \max\{\alpha \in \Gamma_0 \,|\, (\alpha,\vec{k}) \in \Gamma_0 \times \vec{J_0} \text{ with } \partial^{(\alpha,\vec{k})} \notin \mathcal{D}_1\}. \tag{4.11}$$

By this definition, we have

$$\partial^{(\alpha,\vec{i})} \notin \mathcal{D}_1 \iff (\alpha,\vec{i}) = (\tau,\vec{k}) \text{ or } \vec{i} <' \vec{k} \text{ or } \vec{i} = \vec{k} \text{ but } \alpha < \vec{\tau}. \tag{4.12}$$

For each $p \in \overline{1,\ell_1}$, we take

$$m_p = \sum_{q=p+1}^{\ell_1} \sum_{(\alpha,\vec{i})\in\Gamma_0\times\vec{J_0}} i_q, \tag{4.13}$$

and $\alpha^{(p)} \in \Gamma$ such that

$$\alpha^{(p)} > m_p\alpha^{(p+1)} \ \text{ for } \ p \in \overline{1,\ell_1 - 1} \text{ and } \alpha^{(\ell_1)} > \beta, \tag{4.14}$$

where β is the largest element of Γ_0. Set

$$d' = \prod_{p=1}^{\ell_1} \exp(x^{\alpha^{(p)}}\partial_p)(d). \tag{4.15}$$

Then d' has a term

$$x^{\mu,\vec{k}'}\partial^{(\tau,\vec{k})} \text{ with } \mu = \sum_{p=1}^{\ell_1} k_p\alpha^{(p)} + \tau, \ \vec{k}' = \vec{k} - \sum_{p=1}^{\ell_1}(k_p)_{[p]}, \ \ \partial^{(\vec{\tau},\vec{k})} \notin \mathcal{D}_1, \tag{4.16}$$

where τ and $\vec{k}$ are defined in (4.10) and (4.11), and any term $x^{\alpha',\vec{i}'}\partial^{(\alpha',\vec{i}')}$ appears in d' with $\partial^{(\alpha',\vec{i}')} \notin \mathcal{D}_1$ must be of the form $x^{\nu,\vec{m}}\partial^{(\alpha,\vec{i})}$ with

$$\nu = \sum_{p=1}^{\ell_1} n_p\alpha^{(p)} + \alpha, \ n_p \leq i_p \ \text{ for } \ p \in \overline{1,\ell_1} \text{ and } \vec{m} = \vec{i} - \sum_{p=1}^{\ell_1}(n_p)_{[p]}, \ \partial^{(\alpha,\vec{i})} \notin \mathcal{D}_1. \tag{4.17}$$

Let ν and $\vec{i}$ be as in (4.17). By (4.10) and (4.11), we have the following two cases. First we have $\vec{i} <' \vec{k}$. We let p' be the first index with $i_{p'} < k_{p'}$. By (4.12) and (4.13),

$$\nu \leq \sum_{p=1}^{p'} n_p\alpha^{(p)} + m_{p'}\alpha^{(p'+1)} < \sum_{p=1}^{p'-1} k_p\alpha^{(p)} + k_{p'}\alpha^{(p')} \leq \mu. \tag{4.18}$$

Secondly we have $\vec{i} = \vec{k}$ but $\alpha < \tau$, which implies $\nu < \mu$. Thus we have proved that μ in (4.16) is the element δ defined in (4.6) for d' and $\vec{k}'$ is the corresponding element $\vec{j}$ defined in (4.7) for d', which satisfies (4.8) by (4.16). This proves Claim 1.

Claim 2. Replacing d by $\overline{\psi}(d)$ for some $\overline{\psi}$ defined in (2.31) and (2.32), and considering the derivation $\overline{\psi}(d)$ of $\mathcal{S}'$, we can assume that if a term $x^{\alpha,\vec{i}}\partial_p$ appears in d with $\alpha \geq \delta$ and $p \in \overline{1,\ell_1}$, then

$$p > 1 \text{ and } i_1 = 0, \text{ or } p = 1 \text{ and } i_1 \leq 1, i_q = 0 \text{ for } q \in \overline{2,\ell_1}. \tag{4.19}$$

Pick $\mu, \nu \in \Gamma$ such that

$$\mu > (m_1+1)\nu, \quad \nu > \beta \text{ and } \nu > -\min\{\alpha \,|\, (\alpha,\vec{i}) \in \Gamma_0 \times \vec{J}_0\}, \tag{4.20}$$

where m_1 is defined in (4.13) with $p = 1$. Define $\overline{\psi}$ as in (2.31) and (2.32) with

$$\alpha^{(1)} = -\mu, \ \alpha^{(p)} = -\nu \ \text{ for } \ p \in \overline{2,\ell_1}. \tag{4.21}$$

Then we have

$$\overline{\psi}(x^{\alpha,\vec{i}}\partial_p) = x^{\gamma,\vec{i}}\partial_p \ \text{ with } \ \gamma = \alpha - i_1\mu - \sum_{p=2}^{\ell_1} i_p\nu + \delta_{p,1}(\mu-\nu) + \nu \ \text{ for } \ p \in \overline{1,\ell_1}, \tag{4.22}$$

$$\overline{\psi}(x^{\alpha,\vec{i}}\partial_p) \in \mathcal{S}_\gamma \ \text{ with } \ \gamma = \alpha - i_1\mu - \sum_{p=2}^{\ell_1} i_p\nu \ \text{ for } \ p \in \overline{\ell_1+1,\ell}. \tag{4.23}$$

By (4.22) and (4.23), we see that any term $x^{\alpha,\vec{i}}\partial_p$ appearing in d is mapped to $\mathcal{S}_\gamma$ with some $\gamma < \delta$ if $\vec{i}$ does not satisfy (4.19). On the other hand, if we write $\partial^{(\delta,\vec{j})} = \partial' + \partial''$ with $\partial' \in \mathcal{D}_1$, $\partial'' \in (\mathcal{D}_2+\mathcal{D}_3)\backslash\{0\}$ for the term $x^{\delta,\vec{j}}\partial^{(\delta,\vec{j})}$ in d, then by (2.32), $x^{\delta,\vec{j}}\partial''$ is mapped to an element in $\mathcal{S}_\delta$ with a term $x^{\delta,\vec{j}}\partial''$. Thus we have (4.19) for $\overline{\psi}(d)$.

Now suppose d satisfies (4.8) and (4.19) with $\delta, \vec{j}$ defined by (4.6) and (4.7). If there exists a term $x^{\alpha,\vec{i}}\partial_1$ appearing in d with $\alpha > \delta$ and $i_1 = 1$, then we take δ' to be the largest such α, $\vec{j}$ to be such $\vec{i}$ and take $v = x^\gamma\partial_1$ with $\gamma = 0$. Otherwise we take $\delta' = \delta$ and $v = x^\gamma\partial_1$ with $\gamma \in \Gamma$ such that $\langle \partial^{(\delta,\vec{j})}, \gamma\rangle \neq 0$. Then by (4.19), it is straightforward to verify that

$$d^k(v) \equiv ad_u^k(v) \not\equiv 0 \,(\mathrm{mod}\,\mathcal{S}_{(\gamma+k\delta')}) \ \text{ with } \ u = x^{\delta',\vec{j}}\partial^{(\delta',\vec{j})}, \tag{4.24}$$

where $\mathcal{S}_{(\alpha)} = \bigoplus_{\alpha>\gamma\in\Gamma}\mathcal{S}_\gamma$. Thus d is not locally-finite. This proves the first statement in (4.5), and the second can be proved similarly. $\square$

An element $u \in \mathcal{W}$ is called *locally-nilpotent* if ad_u is a locally-nilpotent. If $\ell_1 \geq 1$ in the proof of Lemma 4.1, then we can always take locally-nilpotent $v \in \mathcal{AD}_1$ in (4.24). Denote

$$\mathrm{Nil}(\mathcal{S}) = \text{ the subalgebra generated by locally-nilpotent elements in } \mathcal{S}\cap\mathcal{AD}_1. \tag{4.25}$$

Then we have the following lemma.

Lemma 4.2. *Suppose $\ell_1 \geq 1$. If $d \in \mathrm{Der}\,\mathcal{S}$ is locally-nilpotent on* $\mathrm{Nil}(\mathcal{S})$, *then* $d \in \mathcal{AD}_1 \oplus \mathcal{D}^-$. $\square$

Set

$$A_1 = \mathrm{Span}\{x^\alpha \,|\, \alpha \in \Gamma\}, \quad A_0 = \mathbb{F}[t_1, t_2, ..., t_{\ell_1+\ell_2}]. \tag{4.26}$$

We have $\mathcal{A} = A_0 \cdot A_1$.

Lemma 4.3. *Let $u = x^\alpha \in A_1$, $d = \bar{x}^{\beta,\vec{j}}\partial \in \mathcal{S}_\beta \cap \mathcal{AD}_1$ be homogeneous elements.*
(i) If ad_d *is locally-nilpotent on $\mathcal{S} \cap \mathcal{AD}_1$, then* ad_{ud} *is locally-nilpotent on $\mathcal{AD}_1$.*
(ii) If $\alpha + \beta \neq \rho$ or $\ell_2 \geq 1$, then $ud \in \mathcal{S}$.

Proof. (i) Note that for any $v = x^{\beta,\vec{i}}\partial \in \mathcal{AD}_1$, we have

$$(\mathrm{ad}_{ud})^m(v) = u^m(\mathrm{ad}_d)^m(v) \ \ \text{for}\, m \in \mathbb{N}. \tag{4.27}$$

This proves (i).

(ii) Recall that $\mathcal{S}_\beta$ is spanned by $D_{p,q}(x^{\beta,\vec{i}})$ with $\vec{i} \in \mathbb{N}^{\ell_1+\ell_2}$ and $p, q \in \overline{1, \ell}$. Using induction on $\vec{j}$, one can prove that the element d is a linear combination of

$$D_{p,q}(x^{\beta,\vec{i}}) \ \text{ with } \ p, q \in \overline{1, \ell_1} \ \text{ or } \ p \in \overline{1, \ell_1},\, q \in \overline{\ell_1 + 1, \ell} \ \text{ but } \ i_p = 0. \tag{4.28}$$

Thus we can assume that d has the form (4.28). If $p, q \in \overline{1, \ell_1}$, then

$$ud = uD_{p,q}(x^{\beta,\vec{i}}) = D_{p,q}(x^{\alpha+\beta,\vec{i}}) \in \mathcal{S}. \tag{4.29}$$

If $p \in \overline{1, \ell_1}$, $q \in \overline{\ell_1 + 1, \ell}$ and $i_p = 0$, then

$$D_{p,q}(x^{\beta,\vec{i}}) = (\beta_{q-\ell_1} - \rho_{q-\ell_1})x^{\beta,\vec{i}}\partial_p + i_q x^{\beta,\vec{i}-1_{[q]}}\partial_p. \tag{4.30}$$

Assume $\alpha + \beta \neq \rho$. Choose $q' \in \overline{\ell_1 + 1, \ell}$ with $a = \alpha_{q'-\ell_1} + \beta_{q'-\ell_1} - \rho_{q'-\ell_1} \neq 0$. Then

$$\begin{aligned} & ud - (\beta_{q-\ell_1} - \rho_{q-\ell_1})a^{-1}D_{p,q'}(x^{\alpha+\beta,\vec{i}}) \\ = \ & i_q x^{\alpha+\beta,\vec{i}-1_{[q]}}\partial_p - (\beta_{q-\ell_1} - \rho_{q-\ell_1})a^{-1} i_{q'} x^{\alpha+\beta,\vec{i}-1_{[q']}}\partial_p. \end{aligned} \tag{4.31}$$

by induction on $\vec{i}$, we prove (ii). The case $\alpha + \beta = \rho$ and $\ell_2 \geq 1$ can be proved similarly. $\square$

Theorem 4.4 (Main Theorem). *The Lie algebras $\mathcal{S} = \mathcal{S}(\ell_1, \ell_2, \ell_3; \rho, \Gamma)$ with $\ell \geq 3$ and $\mathcal{S}' = \mathcal{S}(\ell'_1, \ell'_2, \ell'_3; \rho', \Gamma')$ (cf. (1.26)) are isomorphic if and only if $(\ell_1, \ell_2, \ell_3) = (\ell'_1, \ell'_2, \ell'_3)$ and there exists an element $g \in G_{\ell_2,\ell_3}$ (cf. (1.18)) such that $g(\Gamma) = \Gamma'$, and $g(\rho) = \rho'$ (cf. (1.19)) if $\ell_1 = 0$. In particular, there exists a one-to-one correspondence between the set of isomorphism classes of the Lie algebras of the form (1.26) and the set*

SW in (1.23) if $\ell_1 > 0$, *and between the set of isomorphism classes of the Lie algebras of the form (1.26) and the set SS defined in (1.28) if* $\ell_1 = 0$.

Proof. We shall use the notations with a prime to denote elements and vector spaces related to $\mathcal{S}'$; for instance, $\mathcal{A}'$, $x'^{\alpha',\vec{i}}$. Let $\sigma : \mathcal{S} \to \mathcal{S}'$ be a Lie algebra isomorphism. Then σ induces an isomorphism, also denoted by σ, of $\mathrm{Der}\,\mathcal{S}$ to $\mathrm{Der}\,\mathcal{S}'$. For convenience, we view $\mathcal{S}$ as a Lie subalgebra of $\mathrm{Der}\,\mathcal{S}$ by identifying $u \in \mathcal{S}$ with ad_u when the context is clear.

<u>*Case 1*</u>. $\ell_1 = 0$.

By Lemma 4.1, the maximal space of locally-finite inner derivations of $\mathcal{S}$ is $\mathcal{D}_\rho$ which is of finite dimensional. So any maximal space of locally-finite inner derivations of $\mathcal{S}'$ is of finite dimensional. Again by Lemma 4.1, we must have

$$\ell_1' = 0 = \ell_1 \quad \text{and} \quad \dim \mathcal{D}_\rho = \dim \mathcal{D}'_{\rho'}. \tag{4.32}$$

If $\ell_2 = 0$, then by Lemma 4.1, $\mathcal{S}$ has no locally-nilpotent derivations and so does $\mathcal{S}'$. Hence $\ell_2' = 0$ by (4.3). Our theorem follows from Theorem 5.1 in [DZ]. Now we assume $\ell_2, \ell_2' \geq 1$.

By Lemma 4.1, $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$, $\mathcal{D}_3 \cap \mathcal{D}_\rho$ and $\mathcal{D}_\rho$ are the space of semi-simple derivations, the space of semi-simple inner derivations and the space of locally-finite inner derivations of $\mathcal{S}$, respectively. Thus

$$\sigma(\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma', \mathbb{F}), \quad \sigma(\mathcal{D}_3 \cap \mathcal{D}_\rho) = \mathcal{D}_3' \cap \mathcal{D}'_{\rho'}, \quad \sigma(\mathcal{D}_\rho) = \mathcal{D}'_{\rho'}. \tag{4.33}$$

Note that the algebra $\mathcal{S} = \bigoplus_{\alpha\in\Gamma} \mathcal{S}_\alpha$ (cf. (2.9)) is a Γ-graded Lie algebra, whose homogeneous components $\mathcal{S}_\alpha$ are precisely the weight spaces of $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{F})$. Thus there exists an additive group isomorphism

$$g : \Gamma \to \Gamma' \;\text{ with }\; g(0) = 0 \;\text{ such that }\; \sigma(\mathcal{S}_\alpha) = \mathcal{S}'_{g(\alpha)} \;\text{ for }\; \alpha \in \Gamma. \tag{4.34}$$

In particular,

$$\mathcal{S}_0 \cong \mathcal{S}_0', \quad (\mathrm{Der}\,\mathcal{S})_0 \cong (\mathrm{Der}\,\mathcal{S}')_0. \tag{4.35}$$

We want to prove that for any $\alpha \in \Gamma$ and $\partial \in \mathcal{D}_{\alpha-\rho}$, there exists a unique $\partial' \in \mathcal{D}'_{\tau(\alpha)-\rho'}$ such that

$$\sigma(x^\alpha \partial) = x'^{\tau(\alpha)} \partial'. \tag{4.36}$$

Suppose that a term $x'^{\tau(\alpha),\vec{i}}\partial''$ appears in $\sigma(x^\alpha\partial)$ with $i_q \neq 0$ for some $q \in \overline{1, \ell_1' + \ell_2'}$. Then

$$[\partial_{t_q'}, \sigma(x^\alpha\partial)] \neq 0. \tag{4.37}$$

Since $\partial_{t_q'}$ is locally-nilpotent derivation on $\mathcal{S}'$, $\sigma^{-1}(\partial_{t_q'})$ is also locally-nilpotent derivation on $\mathcal{S}$. By Lemma 4.1, $\sigma^{-1}(\partial_{t_q'}) \in \mathcal{D}^-$. So we have $[\sigma^{-1}(\partial_{t_q'}), x^\alpha\partial] = 0$, which contradicts

(4.37). Thus (4.36) holds, and σ induces an isomorphism $\sigma : \mathcal{S}^{[0]} \cong \mathcal{S}'^{[0]}$. By Theorem 5.1 in [DZ], σ induces an isomorphism $\phi = \sigma|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}'$ such that

$$g(\rho) = \rho', \ \ \langle \partial, \alpha \rangle = \langle \phi(\partial), g(\alpha) \rangle \ \text{ for } \ \alpha \in \Gamma. \tag{4.38}$$

In particular, $\ell_2 + \ell_3 = \ell_2' + \ell_3'$. Since $\mathcal{D}_3$ is a subspace of semi-simple derivations on $\mathcal{S}$ and $\mathcal{D}_3'$ is the subspace of all semi-simple derivations of $\mathcal{D}'$, we must have $\phi(\mathcal{D}_3) = \mathcal{D}_3'$, and $\ell_3 = \ell_3'$. Moreover, $g \in G_{\ell_2,\ell_3}$ (cf. (1.18) and (1.19)) by the second equation in (4.38).

Case 2. $\ell_1 \geq 2$.

By Case 1, $\ell_1' \geq 1$. By (2.34), we can assume $\rho = \rho' = 0$. Then by (2.13), $\mathcal{D} = \mathcal{D}_0 \in \mathcal{S}$. For $p \in \overline{1, \ell_1}$, since $\partial_p \in \mathcal{S}$ is a locally-nilpotent inner derivation of $\mathcal{S}$, Lemma 4.1 implies $\sigma(\partial_p) \in \mathcal{S}' \cap \mathcal{A}'\mathcal{D}_1'$. Write

$$\sigma(\partial_p) = \sum_{\alpha' \in \Gamma_p'} u_{\alpha'}', \qquad \text{where } \ u_{\alpha'}' \in \mathcal{A}_{\alpha'}'\mathcal{D}_1' \tag{4.39}$$

and Γ_p' is a finite subset of Γ'. By Lemma 4.3 (ii), we have

$$x'^{\beta'}\sigma(\partial_1) \in \mathcal{S}' \qquad \text{for } \beta' \in \Gamma' \backslash (-\Gamma_p'), \tag{4.40}$$

where $-\Gamma_p' = \{-\alpha' \mid \alpha' \in \Gamma_p'\}$. Set

$$B' = \{z \in \mathcal{A}' \mid z\sigma(\partial) \in \mathrm{Der}\,\mathcal{S}' \text{ for } \partial \in \mathcal{D}_1\}. \tag{4.41}$$

Let

$$\overline{A} \text{ be the subalgebra of } \mathcal{A}' \text{ generated by } B'. \tag{4.42}$$

Since $\Gamma_0' = -\cup_{p=1}^{\ell_1} \Gamma_p'$ is a finite subset of Γ' and Γ' is a torsion-free group, Γ' is generated by $\Gamma' \backslash \Gamma_0'$. Since A_1' (cf. (4.26)) is the group algebra $\mathbb{F}[\Gamma']$, (4.40) shows

$$A_1' \subset \overline{A}. \tag{4.43}$$

For any $z \in B'$ and $p \in \overline{1, \ell}$, we have

$$\sigma(\partial_p)(z)\sigma(\partial) = [\sigma(\partial_p), z\sigma(\partial)] = \sigma([\partial_p, \sigma^{-1}(z\sigma(\partial))]) \qquad \text{for } \ \partial \in \mathcal{D}_1. \tag{4.44}$$

Thus B' is $\sigma(\mathcal{D})$-invariant, and so is $\overline{A}$. As ∂_p is locally-nilpotent or locally-finite on $\mathcal{S}$ for $p \in \overline{1, \ell_1}$ or $p \in \overline{1, \ell}$, (4.44) implies that $\sigma(\partial_p)$ is locally-nilpotent or locally-finite on $\overline{A}$. By these facts, we have

$$\overline{A} = \bigoplus_{\beta \in \mathbb{F}^{\ell_2+\ell_3}} \overline{A}_\beta, \tag{4.45}$$

with

$$\overline{A}_\beta = \{z \in \mathcal{A}' \mid (\sigma(\partial_p) - \overline{\beta}_p)^m(z) = 0 \text{ for } p \in \overline{\ell_1 + 1, \ell} \text{ and some } m \in \mathbb{N}\}, \tag{4.46}$$

where we have written $\beta=(\overline{\beta}_{\ell_1+1},\cdots,\overline{\beta}_\ell)$. Set

$$\overline{A}_\beta^{(0)}=\{z\in\overline{A}_\beta\,|\,\sigma(\partial_p)(z)=0,\sigma(\partial_q)(z)=\overline{\beta}_q z,\ \ \text{for } p\in\overline{1,\ell_1},\, q\in\overline{\ell_1+1,\ell}\}. \tag{4.47}$$

Obviously

$$\overline{A}_\beta\neq\{0\}\Longleftrightarrow\overline{A}_\beta^{(0)}\neq\{0\}. \tag{4.48}$$

Take

$$\overline{\Gamma}=\{\beta\in\mathbb{F}^{\ell_2+\ell_3}\,|\,\overline{A}_\beta^{(0)}\neq\{0\}\}. \tag{4.49}$$

Since B' is $\sigma(\mathcal{D})$-invariant, we have

$$B'=\bigoplus_{\beta\in\overline{\Gamma}}B'_\beta,\ B'_\beta\neq\{0\}\Longleftrightarrow B'^{(0)}_\beta\neq\{0\},\ B'_\beta=B'\cap\overline{A}_\beta,\ B'^{(0)}_\beta=B'\cap\overline{A}_\beta^{(0)}. \tag{4.50}$$

Set

$$\overline{\Gamma}'=\{\beta\in\overline{\Gamma}\,|\,B'^{(0)}_\beta\neq\{0\}\}. \tag{4.51}$$

Since $\overline{A}$ is generated by B' and

$$B'^{(0)}_\beta B'^{(0)}_\gamma\subseteq\overline{A}^{(0)}_{\beta+\gamma},\ \ \overline{A}_\alpha=\sum_{\beta+\gamma=\alpha}B'_\beta B'_\gamma\quad\text{for } \beta,\gamma\in\overline{\Gamma}',\,\alpha\in\overline{\Gamma}, \tag{4.52}$$

we have

$$\overline{\Gamma}=\ \text{the group generated by } \overline{\Gamma}'. \tag{4.53}$$

For any $\partial\in\mathcal{D}_1\backslash\{0\}$, there exists $z\in B'$ such that $z\sigma(\partial)\notin\sigma(\mathcal{D})$ because

$$\mathrm{Span}\{x'^{\alpha'}\sigma(\partial)\,|\,\alpha'\in\Gamma'\backslash(-\Gamma'_0)\}\ \text{ is infinite-dimensional.} \tag{4.54}$$

Write

$$z\sigma(\partial)=\sigma(w),\ w=\sum_{\alpha\in G}x^\alpha u_\alpha,\ u_\alpha=\sum_{\vec{i}\in J_\alpha}t^{\vec{i}}\partial^{\alpha,\vec{i}}, \tag{4.55}$$

where G is the finite subset of the elements $\alpha\in\Gamma$ such that $u_\alpha\neq 0$ and J_α is the finite subset of the elements $\vec{i}\in\mathbb{N}^{\ell_1+\ell_2}$ such that $\partial^{\alpha,\vec{i}}\in\mathcal{D}\backslash\{0\}$. Moreover, $G\neq\{0\}$ or $J_0\neq\{0\}$. We claim that

$$w\in\mathcal{A}\mathcal{D}_1. \tag{4.56}$$

If not, by Lemma 4.2, there exists $u\in\mathrm{Nil}(\mathcal{S})$ such that ad_w is not nilpotent acting on u. Since $\sigma(u)$ is locally-nilpotent, $\sigma(u)\in\mathcal{S}'\cap\mathcal{A}'\mathcal{D}'_1$ by Lemma 4.1. But by Lemma 4.3 (i), $\mathrm{ad}_{z\sigma(\partial_1)}$ is locally-nilpotent on $\mathcal{A}'\mathcal{D}'_1$, in particular, on $\sigma(u)$. This leads a contradiction.

Note that for any basis $\{d_1,\cdots,d_{\ell_1}\}$ of $\mathcal{D}_1$, there exists an automorphism ι of $\mathcal{S}$ such that

$$\iota(d_p)=\partial_p\ \ \text{for } p\in\overline{1,\ell_1}. \tag{4.57}$$

Assume $G=\{0\}$ in (4.55). Then there exists $\vec{i}\in J_0\backslash\{0\}$, say, $i_p\neq 0$ with $p\in\overline{1,\ell_1+\ell_2}$. By (4.57), we can assume $\partial=\partial_1$ in (4.55). Then for any $\beta\in\Gamma$, we can either take

$u = x^\beta\partial_p$ if $\rho_{p-\ell_1} = 0$, or $u = \rho_{p-\ell_1}x^{\beta,1_{[2]}}\partial_2 - x^\beta\partial_p = D_{2,p}(x^{\beta,1_{[2]}}) \in \mathcal{S}$ if $\rho_{p-\ell_1} \neq 0$, such that

$$\sigma(u)(z)\sigma(\partial) = [\sigma(u), z\sigma(\partial)] = \sigma([u,w]) \neq 0. \tag{4.58}$$

Replacing z by $\sigma(u)(z)$ in (4.55), we can assume $G \neq \{0\}$. Note that for any $\tilde{\partial} \in \mathcal{D}$, we have

$$\sigma(\tilde{\partial})^m(z)\sigma(\partial) = \mathrm{ad}^m_{\sigma(\tilde{\partial})}(z\sigma(\partial)) = \sigma(\sum_{\alpha\in G} x^\alpha \sum_{k=0}^m \binom{m}{k} \alpha(\tilde{\partial})^k\tilde{\partial}^{m-k}(u_\alpha)). \tag{4.59}$$

As $\tilde{\partial}$ is nilpotent on u_α, we obtain that there exists some $z \in B'\backslash\{0\}$ such that

$$z\sigma(\partial) = \sigma(x^\alpha d) \text{ for some } x^\alpha d \in \mathcal{S} \text{ with } \alpha \in \Gamma\backslash\{0\}, d \in \mathcal{D}_1\backslash\{0\} \tag{4.60}$$

by induction on the leading degree of w. Thus

$$\sigma(\tilde{\partial})(z)\sigma(\partial) = [\sigma(\tilde{\partial}), z\sigma(\partial)] = \sigma([\tilde{\partial}, x^\alpha d]) = \alpha(\tilde{\partial})z\sigma(\partial). \tag{4.61}$$

which implies $z \in B'^{(0)}_\alpha$. So $\alpha \in \overline{\Gamma}'$.

We claim that for any $\beta \in \Gamma$, there exists $z' \in B'^{(0)}_\beta\backslash\{0\}$ such that

$$z\sigma(\partial) = \sigma(x^\beta d) \;\text{ for some }\; d \in \mathcal{D}_1\backslash\{0\}. \tag{4.62}$$

Take $\partial = \partial_1$. First suppose $d \notin \mathbb{F}\partial_1$ in (4.60), say, $\chi_p(d) = 1$ for some $p \in \overline{2,\ell_1}$ (cf. (2.26)). Then for any $\beta \in \Gamma$, we have

$$\sigma(x^{\beta-\alpha,1_{[p]}}\partial_1)(z)\sigma(\partial) = [\sigma(x^{\beta-\alpha,1_{[p]}}\partial_1), z\sigma(\partial)] = \sigma([x^{\beta-\alpha,1_{[p]}}\partial_1, x^\alpha d]) = -\sigma(x^\beta\partial_1), \tag{4.63}$$

where the last equality follows from the fact that $d \in \mathcal{D}_1$; that is, (4.62) holds. Next suppose $d = \partial_1$ in (4.60). We choose p with $\alpha_{p-\ell_1} \neq 0$. For any $\beta \in \Gamma$ with $a = \beta_{p-\ell_1} - \alpha_{p-\ell_1} \neq 0$, as in (4.63), we have

$$\begin{aligned}\sigma(ax^{\beta-\alpha,1_{[2]}}\partial_2 - x^{\beta-\alpha}\partial_p)(z)\sigma(\partial) &= [\sigma(ax^{\beta-\alpha,1_{[2]}}\partial_2 - x^{\beta-\alpha}\partial_p), z\sigma(\partial)] \\ &= -\alpha_{p-\ell_1}\sigma(x^\beta\partial_1).\end{aligned} \tag{4.64}$$

Since (4.64) holds for all β with $\beta_{p-\ell_1} \neq \alpha_{p-\ell_1}$, we can derive (4.62) from (4.64). This proves (4.62), and (4.62) proves $\Gamma \subset \overline{\Gamma}'$. Conversely, for any $\beta \in \overline{\Gamma}'$ and $z \in B'^{(0)}_\beta$, (4.59) implies $\beta \in \Gamma$, which together with (4.53) proves

$$\Gamma = \overline{\Gamma}' = \overline{\Gamma}. \tag{4.65}$$

By (4.62), for any $z \in B'^{(0)}_\alpha\backslash\{0\}$ and $\alpha \in \overline{\Gamma}'$, we can write

$$z\sigma(\partial) = \sigma(x^\alpha\tau_z(\partial)) \;\text{ with }\; \tau_z(\partial) \in \mathcal{D}_1 \;\text{ for }\; \partial \in \mathcal{D}_1. \tag{4.66}$$

Hence we get an injective linear transformation τ_z on $\mathcal{D}_1$ because $\mathcal{A}'$ has no zero divisors. Since $\mathcal{D}_1$ is finite dimensional, τ_z is a linear automorphism. For any $\alpha \in \Gamma$ and $z \in {B'}^{(0)}_\alpha \setminus \{0\}$, we claim that τ_z is a scalar operator. Otherwise by (4.57), we can assume

$$z\sigma(\partial_1) = \sigma(x^\alpha \partial_2). \tag{4.67}$$

Take an eigenvector d of τ_z. By (4.67), we see that $\partial_1, \partial_2, d$ are linearly independent since $\mathcal{A}'$ has no zero divisors. So we can assume $d = \partial_3$, that is,

$$z\sigma(\partial_3) = \sigma(x^\alpha \partial_3). \tag{4.68}$$

Applying $\sigma(t_2\partial_1)$ to (4.67) and (4.68), we obtain that $\sigma(t_2\partial_1)(z)$ is nonzero and zero, respectively. This leads a contradiction. Thus for any $\alpha \in \Gamma$ and $z \in {B'}^{(0)}_\alpha \setminus \{0\}$, there exists $c_z \in \mathbb{F}\setminus\{0\}$ such that

$$z\sigma(\partial) = c_z \sigma(x^\alpha \partial) \qquad \text{for } \partial \in \mathcal{D}_1. \tag{4.69}$$

In particular,

$$\dim {B'}^{(0)}_\alpha = 1 \qquad \text{for } \alpha \in \Gamma, \tag{4.70}$$

and

$$\overline{A}_\alpha = {B'}^{(0)}_\alpha \overline{A}_0 \qquad \text{for } \alpha \in \Gamma. \tag{4.71}$$

For given $\alpha \in \Gamma$, we denote

$$\tilde{\partial}_1 = x^\alpha \partial_1,\ \ \tilde{\partial}_2 = x^{-\alpha}\partial_2,\ \ \tilde{\partial}_p = \partial_p,\ \ \tilde{\partial}_q = \partial_q + \alpha_q(t_1\partial_1 - t_2\partial_2) \tag{4.72}$$

for $p \in \overline{3, \ell_1}$ and $q \in \overline{\ell_1{+}1, \ell}$. By (2.31) and (2.32), there exists an automorphism of $\mathcal{S}$ which maps ∂_p to $\tilde{\partial}_p$ for $p \in \overline{1, \ell}$ and fixes A_1. Thus we can replace ∂ by $\tilde{\partial}$ in the above arguments. In particular, for any nonzero $z \in {B'}^{(0)}_\alpha$, we let $\tilde{z} \in \tilde{B'}^{(0)}_{-\alpha} \setminus \{0\}$, where $\tilde{B'}^{(0)}_{-\alpha}$ is defined as in (4.41) and (4.50) with respect to $\tilde{\mathcal{D}}_1 = \{\tilde{\partial}_p \,|\, p \in \overline{1, \ell_1}\}$. Moreover, we have

$$\tilde{z}z\sigma(\partial_1) = \tilde{z}c_z\sigma(x^\alpha\partial_1) = c_z\tilde{z}\sigma(\tilde{\partial}_1) = c_z c_{\tilde{z}}\sigma(x^{-\alpha}\tilde{\partial}_1) = c_z c_{\tilde{z}}\sigma(\partial_1) \qquad \text{for } \partial \in \mathcal{D}_1. \tag{4.73}$$

Thus $\tilde{z}z = c_z c_{\tilde{z}} \in \mathbb{F}\setminus\{0\}$. This shows that z is invertible. By the proof of Theorem 2.1 in [SXZ],

$$\Big(\bigcup_{\alpha\in\Gamma} {B'}^{(0)}_\alpha\Big)\setminus\{0\} = \text{ the set of all invertible elements in } B'. \tag{4.74}$$

Note that

$$\text{the set of all invertible elements in } \mathcal{A}' = \Big(\bigcup_{\alpha'\in\Gamma'} \mathbb{F}{x'}^{\alpha'}\Big)\setminus\{0\}. \tag{4.75}$$

Since all but a finite number of ${x'}^{\alpha'}$ are in B', the set in (4.74) must be equal to the set in (4.75); equivalently, there exists a group isomorphism $g : \Gamma \to \Gamma'$ such that

$${B'}^{(0)}_\alpha = \mathbb{F}{x'}^{\tau(\alpha)} \qquad \text{for } \alpha \in \Gamma. \tag{4.76}$$

Thus

$$B'^{(0)} = \bigoplus_{\alpha\in\Gamma} B'^{(0)}_{\alpha} \tag{4.77}$$

is a group algebra. Furthermore, by (4.71),

$$\overline{A} = B'^{(0)}\overline{A}_0 \cong B'^{(0)} \otimes \overline{A}_0. \tag{4.78}$$

By (4.76), we can assume

$$x'^{g(\alpha)}\sigma(\partial) = \sigma(x^{\alpha}\partial) \ \text{ for } \alpha\in\Gamma,\, \partial\in\mathcal{D}_1. \tag{4.79}$$

For any $\partial\in\mathcal{D}_2+\mathcal{D}_3$, $\sigma(\partial)\in\mathcal{A}'\mathcal{D}'_1+\mathcal{D}'$. Define $\phi:\mathcal{D}_2+\mathcal{D}_3\to\mathcal{D}'_2+\mathcal{D}'_3$ by

$$\phi(\partial) = \text{ the image of } \sigma(\partial) \text{ under the projection: } \mathcal{A}'\mathcal{D}'_1\oplus(\mathcal{D}'_2+\mathcal{D}'_3)\to(\mathcal{D}'_2+\mathcal{D}'_3). \tag{4.80}$$

Then by definition,

$$\langle\phi(\partial),\alpha'\rangle x'^{\alpha'}\sigma(\partial_1) = [\sigma(\partial), x'^{\alpha'}\sigma(\partial_1)] = \sigma(\partial)(x'^{\alpha'})\sigma(\partial_1) = \alpha(\partial)x'^{\alpha'}\sigma(\partial_1) \tag{4.81}$$

for $\alpha\in\Gamma$ and $\alpha'=\tau(\alpha)$, where the first equality follows from (4.80) and the last follows from (4.76). Thus we have the second equation of (4.38), by which ϕ must be a bijection because $g:\Gamma\to\Gamma'$ is an isomorphism and Γ,Γ' are nondegenerate. In particular,

$$\ell_2+\ell_3 = \ell'_2+\ell'_3. \tag{4.82}$$

Since an element $\partial\in\mathcal{D}_3$ is semi-simple and the elements of $\mathcal{D}'_2$ are not semi-simple, we must have $\sigma(\partial)\in\mathcal{A}'\mathcal{D}'_1+\mathcal{D}'_3$. Otherwise, $\sigma(\partial)$ is not semi-simple by the proof of Lemma 4.1. Hence

$$\phi(\mathcal{D}_3)\subset\mathcal{D}'_3, \text{ which implies } \ell_3\leq\ell'_3. \tag{4.83}$$

Set

$$B'^{(1)}_0 = \{z\in B'_0 \,|\, \sigma(\mathcal{D})(v)\subset\mathbb{F}\}\subset\overline{A}_0. \tag{4.84}$$

By the proof of Theorem 2.1 in [SXZ], (4.77) and (4.78), we have

$$\begin{aligned} \dim(B'^{(1)}_0/\mathbb{F}) &\leq \text{ the transcendental degree of } \overline{A}_0 \text{ over } \mathbb{F} \\ &\leq \text{ the transcendental degree of } A'_0 \text{ over } \mathbb{F} = \ell'_1+\ell'_2 \end{aligned} \tag{4.85}$$

(cf. (4.26)). We claim that for any $q\in\overline{1,\ell_1+\ell_2}$, there exists $z_q\in B'^{(1)}_0$ such that

$$z_1\sigma(\partial_2) = \sigma(t_1\partial_2),\ \ z_q\sigma(\partial_1) = \sigma(t_q\partial_1) \qquad \text{for } q\in\overline{2,\ell_1+\ell_2}. \tag{4.86}$$

Assume that (4.86) holds. Applying $\sigma(\partial_p)$, $p\in\overline{1,\ell}$ to (4.86), we obtain

$$\sigma(\partial_p)(z_q) = \delta_{p,q} \ \text{ for } p\in\overline{1,\ell},\, q\in\overline{1,\ell_1+\ell_2}, \tag{4.87}$$

which implies that $\{1, z_q \,|\, q \in \overline{1, \ell_1+\ell_2}\}$ is an linearly independent subset of ${B'}_0^{(1)}$. Thus $\ell_1+\ell_2 \le (\dim {B'}_0^{(1)}/\mathbb{F}) \le \ell_1'+\ell_2'$. This together with (4.82), (4.83) gives $\ell_1' \ge \ell_1 \ge 2$. Hence we can exchange positions of $\mathcal{S}$ and $\mathcal{S}'$ so that we obtain $\ell_3' \le \ell_3$ and $\ell_1'+\ell_2' \le \ell_1+\ell_2$, which together with (4.82) and (4.83) imply our theorem in this case.

Now we want to prove (4.86). Choose $\alpha \in \Gamma\setminus\{0\}$ with $\alpha_p \neq 0$ for some p. Let $z \in {B'}_\alpha^{(0)}$ such that

$$z\sigma(\partial_1) = \sigma(x^\alpha \partial_1). \tag{4.88}$$

If $q \in \overline{2, \ell_1}$ or $\alpha_{q-\ell_1} = 0$, then $\alpha_{p-\ell_1} x^{-\alpha, 2_{[q]}} \partial_q + 2x^{-\alpha, 1_{[q]}} \partial_p = D_{p,q}(x^{-\alpha, 2_{[q]}}) \in \mathcal{S}$. Moreover, we have

$$\begin{aligned}
& \sigma(\alpha_{p-\ell_1} x^{-\alpha,2_{[q]}} \partial_q + 2x^{-\alpha,1_{[q]}} \partial_p)(z)\sigma(\partial_1) \\
= \; & [\sigma(\alpha_{p-\ell_1} x^{-\alpha,2_{[q]}} \partial_q + 2x^{-\alpha,1_{[q]}} \partial_p), z\sigma(\partial_1)] \\
= \; & \sigma([\alpha_{p-\ell_1} x^{-\alpha,2_{[q]}} \partial_q + 2x^{-\alpha,1_{[q]}} \partial_p, x^\alpha \partial_1]) \\
= \; & 2\alpha_{p-\ell_1} \sigma(t_q \partial_1).
\end{aligned} \tag{4.89}$$

Thus we have (4.86). If $q \in \overline{\ell_1+1, \ell_1+\ell_2}$ and $\alpha_q \neq 0$, as in (4.89), we have

$$\sigma(\alpha_{q-\ell_1} x^{-\alpha, 1_{[2]}+1_{[q]}} \partial_2 - x^{-\alpha,1_{[2]}} \partial_2 + x^{-\alpha,1_{[q]}} \partial_q)(z)\sigma(\partial_1) = \alpha_{q-\ell_1} \sigma(t_q \partial_1). \tag{4.90}$$

The first equation of (4.86) can be obtained from the second by exchanging positions of q and 1.

<u>*Case 3*</u>. $\ell_1 = 1$.

By Cases 1 and 2, $\ell_1' = 1$. By Lemma 2.3, we can assume $\rho \neq 0 \neq \rho'$. Then by (2.13), $\partial_1 \in \mathcal{D}_\rho \subset \mathcal{S}$. By Lemma 4.1,

$$\{x^{\alpha,\vec{i}} \partial_1 \,|\, (\alpha, \vec{i}) \in \Gamma \times \mathbb{N}^{\ell_1+\ell_2},\, i_1 = 0\} \cup \mathcal{D}^- \tag{4.91}$$

spans the space of locally-nilpotent derivations of $\mathcal{S}$. When $\ell_2 = 0$, $x^\rho \partial_1 \notin \mathcal{S}$, which is a derivation of $\mathcal{S}$ by Lemma 3.3. Since ∂_1 is an inner derivation of $\mathcal{S}$, $\sigma(\partial_1)$ is an inner derivation of $\mathcal{S}'$. Thus

$$\sigma(\partial_1) = \sum_{(\alpha', \vec{i}) \in \Gamma_1' \times J_1'} a_{\alpha', \vec{i}} {x'}^{\alpha', \vec{i}} \partial_1, \tag{4.92}$$

where $a_{\alpha', \vec{i}} \in \mathbb{F}$ and $\Gamma_1' \times J_1'$ is a finite subset of $\Gamma' \times \mathbb{N}^{\ell_1'+\ell_2'}$ such that $i_1 = 0$ for all $\vec{i} \in J_1'$. We have

$${x'}^{\beta'} \sigma(\partial_1) \in \operatorname{Der} \mathcal{S}' \qquad \text{for } \beta' \in \Gamma' \tag{4.93}$$

by Lemma 4.3 (ii) if $\ell_2' \ge 1$. If $\ell_2' = 0$, then $J_1' = \{0\}$. When $\beta' + \alpha' = \rho'$ for some $\alpha' \in \Gamma_1'$, ${x'}^{\beta'} \cdot {x'}^{\alpha'} \partial_1' \in {\mathcal{W}'}_\rho^{[0]} \subset \operatorname{Der} \mathcal{S}'$. By the arguments analogous to those given in Case 2 after (4.40), we can prove the theorem in this case.

This completes the proof of the theorem. □

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